Nievski Seminar.

The Construction of Light.

Il grande pensatore é grande perché é capace di ascoltare l'opera degli altri “grandi” traendone ció che vi é di piú grande e trasformandolo in modo originale.

Martin Heidegger
Nietzsche

Chapter 1

Preliminaries test so good!

Let X be a topological space. $A \subseteq X$ is called *nowhere dense* if $int\overline{A} = \emptyset$. $A \subseteq X$ is called of *1th Baire category* is there exist a sequence $(A_n)_n$ of nowhere dense subsets of X such that

$$A = \bigcup_{n\in\omega} A_n.$$

$A \subseteq X$ is called of *2th Baire category* is A is not of 1th Baire category.

Definition 1.1. A topological space X is called a *Baire space* is every non-empty open of X is of 2th Baire category.

Remark 1.2. It is easy to see that

1. A is nowhere dense in $X \Longleftrightarrow \overline{A}$ is nowhere dense.
2. A is of 1th Baire category in $X \Longleftrightarrow \overline{A}$ is of 1th Baire category.
3. A closed C subset of a topological space X is 1th Baire category if and only if it is countable union of closed nowhere dense.

Proof. For 2. it is enough to note that $\overline{A} = A \cup FrA$, (where FrA is the *boundary* of A) and $intFrA = \emptyset$.

For 3. it is enough to note that if $(K_n)_n$ is a sequence of nowhere dense such that $C = \bigcup_n K_n$ then

$$C = (\bigcup_n \overline{K_n}) \cup FrC.$$

□

Proposition 1.3. *Let (X, τ) be a topological space, A an open subset of X. Then A is 2th Baire category in X if and only if A is 2th Baire category in (A, τ).*

Proof. Easy. □

Theorem 1.4. *(Baire) Let (X, τ) be a topological space. Then the following are equivalent*

(i) *(X, τ) ia a Baire space;*

(ii) *for every family $(A_n)_n$ of open dense subsets of X, then $\bigcap_n A_n$ is dense in X.*

Proof. $(i) \to (ii)$ Suppose that there exists $(A_n)_n$ of open dense subsets of X such that $\bigcap_n A_n$ is not dense in X. Therefore, there exists an open set A such that $A \cap \bigcap_n A_n = \emptyset$. Since, for each $n \in \omega$, A_n is dense, we have that

$$int(A \setminus A_n) = \emptyset$$

and $A \setminus A_n$ is closed. Then $A = \bigcup_n (A \setminus A_n)$ should be an open of 1th Baire category.

$(ii) \to (i)$ Suppose there exists an open of 1th Baire category A. Hence there exists a sequence of nowhere dense $(K_n)_n$ such that

$$A = \bigcup_n K_n.$$

Then $A_n = X \setminus K_n$ is a sequence of open dense of X with $\bigcap_n A_n$ not dense in X (because otherwise we should have $A = \emptyset$). □

Theorem 1.5. *(Baire) Every complete metric space (X, d) is a Baire space.*

Proof. Easy □

Definition 1.6. Let (X, d) be a complete metric space. A function

$$f : X \longrightarrow \mathbb{R}$$

is called of *1th Baire category* if there exists a sequence of continuous functions $(f_n)_n \subseteq C(X)$ such that

$$f(x) = \lim_n f_n(x) \qquad \text{for every } x \in X.$$

We shall denote by $\mathcal{B}_1(X)$ the space of all Baire functions on X, equipped with the pointwise topology.

Let B an open ball of (X, d) and $f : X \longrightarrow \mathbb{R}$ be a function. Let

$$\omega_f(B) = \sup_{x \in B} f(x) - \inf_{x \in B} f(x)$$

$\omega_f(B)$ is called the *oscillation of f in B*.

For any $x \in X$, we define

$$\omega_f(x) = \lim_{\delta \to 0} \omega_f(B(x, \delta)).$$

$\omega_f(x)$ is called the *oscillation of f in x*.

It is clear that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$. Moreover

$$D_f = \bigcup_n \{x \in X : \ \omega_f(x) \geq \frac{1}{n}\}$$

coincides with the set of discontinuity points of f and every $\{x \in X : \ \omega_f(x) \geq \frac{1}{n}\}$ is closed. Then the discontinuity points of a function $f : X \longrightarrow \mathbb{R}$ is a F_σ set.

Theorem 1.7. *(Baire) Let (X, d) be a complete metric space, $f : X \longrightarrow \mathbb{R}$ be a 1th Baire category function. Then f is continuous except a set of points of 1th Baire category.*

Proof. It is enough to show that for every $\varepsilon > 0$

$$F = \{x \in X : \ \omega_f(x) \geq 5\varepsilon\}$$

is nowhere dense.

Let $(f_n)_n$ be a sequence of continuous functions such that $(f_n)_n$ converges pointwise to f. Let

$$E_n = \bigcap_{i,j \geq n} \{x \in X : \ |f_i(x) - f_j(x)| \leq \varepsilon\}.$$

Then

(1) E_n is closed for all $n \in \omega$;

(2) $E_n \subseteq E_{n+1}$ for all $n \in \omega$;

(3) $\bigcup_n E_n = X$.

Since X is a Baire space, for each closed $C \subseteq X$, there exist an open subset A_C of X, $n_0 \in \omega$ such that

$$A_C \subseteq C \cap E_{n_0}.$$

That means

$$|f_i(x) - f_j(x)| \leq \varepsilon \quad \forall x \in A_C,\ i, j \geq n_0.$$

For $j = n$ and $i \to \infty$ we get

$$(1) \qquad |f(x) - f_n(x)| \leq \varepsilon \quad \forall x \in A_C.$$

Now, for each $x_0 \in A_C$, there exists $I(x_0) \subseteq A_C$ neighborhood of x_0 such that

$$(2) \qquad |f_n(x) - f_n(x_0)| \leq \varepsilon \quad \forall x \in I(x_0).$$

Putting (1) and (2) together, we have

$$|f(x) - f_n(x_0)| \leq \varepsilon \quad \forall x \in I(x_0).$$

Therefore $\omega_f(x_0) \leq 4\varepsilon$. So no points of A_C belongs in F. But C was an arbitrary closed such that there exist an open A_C and

$$A_C \subseteq C \setminus F.$$

That implies F is nowhere dense □

Using the fact that a F_σ set is of 1th Baire category if and only if its complement is dense, we get

Corollary 1.8. *Let (X, d) be a complete metric space and $f : X \longrightarrow \mathbb{R}$. Then*

$f \in \mathcal{B}_1(X)$ if and only if f is continuous at a dense set of points.

Corollary 1.9. *(R. Baire, 1899) Let (X, d) be a complete metric space. A function f on X is 1th Baire function if and only if its restriction to every closed subset M of X has a point of continuity.*

Proof. If $D_M = D_f \cap M$ is the set of discontinuity points of f in M, we have that D_M is a F_σ set of 1th Baire category of M. □

1.0.1 The spaces $C_p(X)$ and $\mathcal{B}_1(X)$

Definition 1.10. For a compact topological space X, we denote by $C(X)$ the space of all continuous real-valued functions on X. On such space, we consider

(i) the *norm topology*: the topology defined by the norm

$$\|f\| = \sup_{x\in X} |f(x)|;$$

(ii) the *pointwise topology*: obtained by considering $C(X)$ as a subspace of $\mathbb{R}^X$, the space of all real-valued functions equipped with the product topology. This space is denoted by $C_p(X)$ (X in such case could be a Polish space). A neighborhood of a function f is determined by finite sequence $x_1, \dots, x_n$ of points in X and $\varepsilon > 0$ by

$$U_f(x_1, \dots, x_n, \varepsilon) = \{g \in C_p(X) : \ |g(x_i) - f(x_i)| < \varepsilon, \ \forall i = 1, \dots, n\}$$

Definition 1.11. A space X is *countably compact* iff every sequence in X as a cluster point in X.

For separable metric space this notion is equivalent to compactness, but in general is weaker.

Theorem 1.12. *(Grothendieck) Let X be a compact space and $Y \subseteq C_p(X)$ a closed subspace. Then Y is compact if and only if it is countably compact.*

Proof. Assume Y countably compact. Then, for every $x \in X$ there is a positive real number M_x such that $|f(x)| \leq M_x$ for every $f \in Y$. Since $\overline{Y}$ is a closed subset of $\prod_{x\in X}[-M_x, M_x]$, we have that $\overline{Y}$ taken inside $\mathbb{R}^X$ is compact in $\mathbb{R}^X$.

Claim $\overline{Y}$ lies in $C_p(X)$.

Suppose there exists a discontinuous function $f \in \overline{Y}$. Fix $\varepsilon > 0$ and $y \in X$ such that the set $Z = X \setminus f^{-1}(f(y) - \varepsilon, f(y) + \varepsilon)$ accumulates to y. By induction, we built sequences $\{U_n\}$ of open sets containing y, $(x_n)_n \subseteq Z$ and $(f_n)_n \subseteq Y$ such that

(0) $\overline{U_{n+1}} \subseteq U_n$, for all n;

(1) $|f_n(x) - f_n(y)| < \frac{\varepsilon}{2^n}$ for all $x \in U_n$;

(2) $x_n \in U_n \cap Z$, for all n;

(3) $|f_{n+1}(x_i) - f(y)| > \frac{\varepsilon}{2}$, for $i = 1, \dots, n$;

(4) $|f_n(y) - f(y)| < \frac{\varepsilon}{2^n}$.

Assume that U_i, f_i and x_i are chosen for all $i \leq n$. Then $U_f(x_1, \dots, x_n, y, \frac{\varepsilon}{2}) \cap Y$ is not empty, so pick f_{n+1} in this set. Then

$$|f_{n+1}(x_i) - f(y)| \geq |f(x_i) - f(y)| - |f_{n+1}(x_i) - f(x_i)| > \frac{\varepsilon}{2}.$$

Therefore f_{n+1} satisfy (3). Since f_{n+1} is continuous at y we can pick an open neighborhood U_{n+1} of y such that $\overline{U_{n+1}} \subseteq U_n$ and (1) is satisfied. Finally, for the definition of Z, we can pick $x_{n+1} \in U_{n+1} \cap Z$ to satisfies (1).

Let x_∞ an accumulation point of $(x_n)_n$; in particular $x_\infty \in \bigcap_n \overline{U_n}$. Let $S = (x_n)_n \cup \{x_\infty\}$ and define

$$\Phi : C_p(X) \longrightarrow C_p(S)$$

by

$$\Phi(g) = g|_S$$

Then Φ is continuous. Therefore $F = \Phi(Y)$ is a compact in $C_p(S) \subseteq \mathbb{R}^S$ a separable metric space. Since F is countably compact, we have that F is compact. Let g be an accumulation point of $\{f_n|_S\}_n$. By the construction, we have that $g(x_\infty)$ is not in the closure of $\{g(x_n)\}_n$. Then g is not continuous at x_∞. Namely a contradiction.

□

Let us recall that, if X is a Banach space, for each $x^* \in X^*$ let $D_{x^*} = \mathbb{K}$, and let $\mathcal{D} = \prod_{x^* \in X^*} D_{x^*}$. Let $T : X \longrightarrow \mathcal{D}$ the map defined by

$$T(x) = (x^*(x))_{x^* \in X^*}.$$

Then T is one-to-one embedding of X into $\mathcal{D}$. The *weak topology* on X is defined as the topology induced by $\mathcal{D}$ via the map T. Similarly, we can define on X^* a weaker topology, called the *weak* topology*, which is induced by $\widetilde{\mathcal{D}} = \prod_{x \in X} D_x$, where $D_x = \mathbb{K}$, for each $x \in X$. It is classical, and easy to prove, that the closed unit ball B_{X^*} of X^* is weak* compact (in the literature such a result is called the *Banach-Alaoglu-Boubaki*).

Remark 1.13. Grothendieck's theorem in particular implies:

A sequence $(f_n)_n$ in $(C(X), \|\cdot\|)$ is weakly convergent to a function f if and only if f_n converges pointwise to f.

Definition 1.14. A regular Hausdorff space X is called **angelic space** if

(i) every relatively countably compact is relatively compact;

(ii) for every relatively compact A of X, then $x \in \overline{A}$ if and only if there exists $(x_n)_n \subseteq A$ converging to x.

We notice that even the space $c_0 = C(\alpha\mathbb{N})$, where $\alpha\omega$ is the Alexandroff's compactification of the natural numbers, is not an angelic space.

The next result, via Grothendieck's theorem, tell us that the space $C_p(X)$ is angelic.

Theorem 1.15. *(Eberlain) Let X be a compact space and Y be a compact subset of $C_p(X)$. Then for every $A \subseteq Y$, if f is in the closure $\overline{A}$ of A then there is $(f_n)_n \subseteq A$ converging to f.*

The proof follows by the next two lemmas.

Lemma 1.16. *Under the assumption of the theorem above, there is a countable $A_0 \subseteq A$ such that $f \in \overline{A_0}$*

Proof. Let us assume that $f = \theta_{C_p(X)}$. Fix $n \in \omega$ and $x = (x_1, \dots, x_n) \in X^n$. Pick $f_x \in U_\theta(x_1, \dots, x_n, \frac{1}{n}) \cap A$ and let

$$W_x = \prod_{i=1}^{n} f_x^{-1}(-\frac{1}{n}, \frac{1}{n}).$$

W_x is open in X^n. Since X^n is compact, there exists a finite set $F_n \subseteq X^n$ such that

$$\bigcup_{x \in F_n} W_x = X^n.$$

Let

$$A_0 = \{f_x : \ x \in F_n, \ n \in \omega\}.$$

A_0 is clearly countable. We need to show that $\theta_{C_p(X)} \in \overline{A_0}$.

Given $\varepsilon > 0$ and $x_1, \dots, x_n$. Increasing $n$if needed, we can assume that $\frac{1}{n} \leq \varepsilon$. We need to find $g \in A_0$ such that, for $i = 1, \dots, n$, $|g(x_i)| < \frac{1}{n}$. Choose $y \in F_n$ such that $x = (x_1, \dots, x_n) \in W_y$. then $g = f_y$ works. Indeed, follows form $x_i \in f_y^{-1}(-\frac{1}{n}, \frac{1}{n})$, for $1 \leq i \leq n$, that $|f_y(x_i)| < \frac{1}{n}$. □

Lemma 1.17. *For every countable $A_0 \subseteq Y$, the closure $\overline{A_0}$ is second countable.*

Proof. Let $\Phi : X \longrightarrow \mathbb{R}^{A_0}$ be defined by

$$\Phi(x) = (f(x))_{x \in A_0}.$$

Then Φ is a continuous map. Therefore, $Z = \Phi(X) \subseteq \mathbb{R}^{A_0}$ is a compact second countable space. Let us define

$$\Psi : C_p(Z) \longrightarrow C_p(X)$$

by

$$\Psi(f) = f \circ \Phi.$$

Step 1 Ψ is a homomorphism embedding.

Clearly Ψ is one-to-one.. To see that Ψ is continuous note that

$$\Psi^{-1}(U_{\Phi(f)}(x_1, \ldots, x_n, \varepsilon)) = U_f(\Phi(x_1), \ldots, \Phi(x_n), \varepsilon).$$

On the other hand, for every basic open set $U_f(z_1, \ldots, z_n, \varepsilon)$ of $C_p(Z)$,

$$\Psi(U_f(z_1, \ldots, z_n, \varepsilon)) = U_{\Phi(f)}(x_1, \ldots, x_n, \varepsilon) \cap \Psi(C_p(Z))$$

for every choice of $x_i \in \Phi^{-1}(z_i)$, $i = 1, \ldots, n$. Thus, the inverse of Ψ is also continuous.

Step 2 The range of Ψ is closed in $C_p(X)$.

Take g in the closure of $\Psi(C_p(Z))$ inside $C_p(X)$. For every $z \in Z$, the function g is constant on $\Phi^{-1}(z)$. Otherwise, if for some $x_1, x_2 \in \Phi^{-1}(z)$ the number $\varepsilon = |g(x_1) - g(x_2)|$ is positive, then $U_g(x_1, x_2, \frac{\varepsilon}{4})$ would be a neighborhood of g which doesn't intersect the range of Ψ.

Indeed, if $\widetilde{f} \in U_g(x_1, x_2, \frac{\varepsilon}{4}) \cap Range\Psi$, then $\widetilde{f} = \Psi(f_1)$. But

$$\begin{aligned}\varepsilon = |g(x_1) - g(x_2)| &\leq |g(x_1) - \Psi(f_1(x_1))| + |\Psi(f_1(x_1)) - \Psi(f_1(x_2))| \\ &\quad + |g(x_2) - \Psi(f_1(x_2))| \\ &= |g(x_1) - f_1(z)| + 0 + |g(x_2) - f_1(z)|.\end{aligned}$$

Therefore, either $|g(x_1) - f_1(z)| \geq \frac{\varepsilon}{2}$ or $|g(x_1) - f_1(z)| \geq \frac{\varepsilon}{2}$.

But $|g(x_i) - f_1(z)| = |g(x_i) - \Psi(\widetilde{f}(x_i))|$. That implies $\widetilde{f} \notin U_g(x_1, x_2, \frac{\varepsilon}{4})$.

It follows that there is a function $f : Z \longrightarrow \mathbb{R}$ such that $g = f \circ \Phi$. We need to show that f is continuous.

Let τ the maximal topology on Z for which Φ is continuous. Note that f is τ continuous because

$$\Phi^{-1}(f^{-1}(I)) = g^{-1}(I)$$

is open in X for every rational interval I. Since (Z, τ) is continuous image under Φ, it is compact. But the original topology σ of Z (inherited from $\mathbb{R}^{A_0}$)

is also compact Hausdorff. Since $\sigma \subseteq \tau$ we have that $\sigma = \tau$.This shows that f is continuous. This proves the claim.

It follows that our set A_0 is a subset of the compact set

$$Y \cap \Psi(C_p(Z))$$

so its closure $\overline{A_0}$ is a compact subset of the range of Ψ. Since Ψ is a homomorphism it is enough to show that compact subsets of $C_p(Z)$ are second countable. Recall that a compact space is second countable if and only if there is a countable family of continuous functions which separates its points. Let $D \subseteq Z$ a countable dense of Z. For each $d \in D$ let us consider

$$p_d : C_p(Z) \longrightarrow \mathbb{R}$$

given by

$$p_d(f) = f(d)$$

It is clear that $(p_d)_p$ is a sequence of continuous functions separating the points of Z.

□

Definition 1.18. Let Y be a topological space and f a real-valued function defined on Y. We say that f satisfies the *Discontinuity Criterion* provided there is a non-empty subset $L \subseteq Y$, $r, \delta \in \mathbb{R}$ with $\delta > 0$ so that

$$\text{for every non-empty open } U \subseteq L \text{ (open in } L)$$

$$\exists y, z \in U : \begin{cases} f(y) > r + \delta \\ f(z) < r \end{cases}$$

Proposition 1.19. *Let Y and f as above and suppose f satisfies the Discontinuity Criterion.*

Then there is a closed non-empty subset K of Y such that $f|_K$ has no point of continuity relative to the topological space K.

Suppose moreover that there is a uniformly bounded family F of continuous real-valued functions on Y so that f is in the pointwise closure of F. Then F contains a sequence equivalent in the sup-norm to the usual ℓ_1-basis.

Proof. Let L, r, δ be chosen as in the above definition. Then $K = \overline{L}$ is the desired closed subset.

Now, let us suppose that $f|_L$ is in the pointwise closure of $F|_L$. That means

$$\forall \varepsilon > 0 \ \exists l_1, \ldots, l_n \in L, \ g \in F \ : \ |g(l_i) - f(l_i)| < \varepsilon, \quad i = 1, \ldots n.$$

Step 1 *There exists $(g_n)_n \subseteq F$ such that, if $A_n = \{x \in L : \ g_n(x) > r+\delta\}$ and $B_n = \{x \in L : \ g_n(x) < r\}$, then*

(1) $A_n \cap B_n = \emptyset$ for each $n \in \omega$;

(2) for every finite subsets $F_1, F_2 \subseteq \omega$ with $F_1 \cap F_2 = \emptyset$ we have $\left(\bigcap_{n\in F_1} A_n\right) \cap \left(\bigcap_{n\in F_2} B_n\right) \neq \emptyset$.

For sake of notation, let us denote by $A_i = A_i$ and $-A_i = B_i$.

Indeed, by hypothesis, choose $y_1, y_2 \in L$ such that $f(y_1) > r+\delta$, $f(y_2) < r$. Since f is in the pointwise closure of F, there must exists $g_1 \in F$ such that

$$g_1(y_1) > r + \delta,\ \ g_1(y_2) < r.$$

Trivially, we have (1) and (2) above.

Suppose $g_1, \dots, g_n \in F$ have been chosen so that

$$\bigcap_{i=1}^{n-1} \epsilon_i A_i \neq \emptyset$$

for each choice of signs $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1})$, with $\epsilon_i = \pm 1$.

Since $\bigcap_{i=1}^{n-1} \epsilon_i A_i$ is a non-empty open set in L, by hypothesis we can pick $y_1^\epsilon, y_2^\epsilon \in \bigcap_{i=1}^{n-1} \epsilon_i A_i$ such that

$$f(y_1^\epsilon) > r + \delta, \quad f(y_2^\epsilon) < r.$$

Again, we can choose $g_n \in F$ such that

$$g_n(y_1^\epsilon) > r + \delta, \quad g_n(y_2^\epsilon) < r,$$

for all 2^{n-1} choices of ϵ.

It follows that $(g_n)_n$ satisfies the Step 1.

Step 2 $(g_n)_n$ *is equivalent (in the sup norm) to the usual* ℓ_1 *basis.*

By multiplying all g_n's by -1 we can assume $r + \delta > 0$. Let $(c_i)_i$ be a sequence of scalars only finite many c_i's non zero so that $\sum_i |c_i| = 1$.

It is enough to show that there is an $s \in L$ such that

$$|\sum_i c_i g_i(s)| \geq \frac{\delta}{2}.$$

Indeed, by homogeneity we get

$$\frac{\delta}{2} \sum_i |c_i| \leq \|\sum_i c_i g_i\| \leq \sum_i |c_i|,$$

which means that $(g_n)_n$ is equivalent to the ℓ_1 basis.

Let $G = \{i \in \omega : \ c_i > 0\}$ and $B = \{i \in \omega : \ c_i < 0\}$. By (2) of Step 1, we can choose

$$(*) \qquad x \in \left(\bigcap_{i\in G} A_i\right) \cap \left(\bigcap_{i\in B} B_i\right), \quad y \in \left(\bigcap_{i\in B} A_i\right) \cap \left(\bigcap_{i\in G} B_i\right).$$

If we suppose first $r \geq 0$, setting $B' = \{i \in B : \ g_i(x) > 0\}$ then

$$\sum_{i\in B} c_i g_i(x) \geq \sum_{i\in B'} c_i g_i(x) > -r \sum_{i\in B'} |c_i| \geq \sum_{i\in B} |c_i|(-r).$$

Similarly

$$-\sum_{i\in B} c_i g_i(y) \geq \sum_{i\in B} |c_i|(-r)$$

For $(*)$ then we have

$$(a) \qquad \sum_i c_i g_i(x) \geq \sum_{i\in G} |c_i|(\delta + r) + \sum_{i\in B} |c_i|(-r)$$

and

$$(b) \qquad -\sum_i c_i g_i(y) \geq \sum_{i\in B} |c_i|(\delta + r) + \sum_{i\in G} |c_i|(-r).$$

Actually, the inequality (a) and (b) hold for $r < 0$ too.

Therefore

$$\begin{aligned}
&\sum_i |c_i| g_i(x) - \sum_i |c_i| g_i(y) \geq \\
&\qquad \geq \sum_{i\in G} |c_i|(\delta + r) + \sum_{i\in B} |c_i|(-r) + \sum_{i\in B} |c_i|(\delta + r) + \sum_{i\in B} |c_i|(-r) \\
&\qquad = \sum_{i\in G} |c_i|\delta + \sum_{i\in B} \delta \\
&\qquad = \delta.
\end{aligned}$$

That implies

$$\text{either } \sum_i |c_i| g_i(x) \geq \frac{\delta}{2} \text{ or } -\sum_i |c_i| g_i(y) \geq \frac{\delta}{2}.$$

In any case, $s = x$ or $s = y$ satisfies the conclusion. □

Lemma 1.20. *Let X be a Polish space and let $(f_n)_n$ be a pointwise bounded sequence of real valued functions on X such that $(f_n)_n$ has no pointwise convergent subsequence.*

Then, there are $N' \subseteq \omega$ and real numbers r, δ with $\delta > 0$ so that for every $M \subseteq N'$ there is $x \in X$ such that

$$(1) \quad f_m(x) > r + \delta \quad \text{for infinitely many } m \in M$$

and

$$f_m(x) < r \quad \text{for infinitely many } m \in M.$$

Proof. Suppose not. Let us enumerate $\mathbb{Q} \times \mathbb{Q}$ by $\{(r_n, \delta_n)\}_n$.

Let $M_0 = \omega$. We now choose infinite sets $M_0 \supseteq M_1 \supseteq \ldots \supseteq M_n \supseteq \ldots$ as follows: suppose M_{n-1} has been already chosen, since (1) above is false, then there exists $M_n \subseteq M_{n-1}$ so that every $x \in X$ fails to satisfies (1) for M_n and (r_n, δ_n).

By a diagonalization argument, we can choose $M \subseteq_a M_n \ \forall n \in \omega$ such that for every $x \in X$ does not exist $(r, \delta) \in \mathbb{Q} \times \mathbb{Q}$ satisfying (1).

But $(f_n)_{n \in M}$ is pointwise bounded and non converging sequence, then there exists $x \in X$ such that

$$\liminf_{m \in M} f_m(x) \lneq \limsup_{m \in M} f_m(x).$$

Now, simply choose rational numbers r, δ with $\delta > 0$ such that

$$\liminf_{m \in M} f_m(x) < r < r + \delta < \limsup_{m \in M} f_m(x).$$

Therefore x satisfies (1) with M r and δ. Namely a contradiction. □

Theorem 1.21. *Let X be a Polish space and let $(f_n)_n$ be a pointwise bounded sequence of real valued functions on X, such that $(f_n)_n$ has no pointwise convergent subsequence. Then there exists a non empty subset $L \subseteq X$ and a subsequence $(f_{n_k})_k$ which is pointwise convergent on L so that the limit function f satisfies the Discontinuity Criterion.*

Consequently, $(f_{n_k})_k$ has no 1th Baire class cluster point in the topology of pointwise convergence.

Proof. Let N', r, δ as the lemma above.

For every $M \subseteq N'$ let $K(M)$ the closure of the set of all $x \in X$ satisfying (1) of the previous lemma. We have

(a) $K(M)$ is a non empty closed set of X, for each $M \subseteq N'$;

(b) $K(M_1) \subseteq K(M_2)$ whenever $M_1 \subseteq_a M_2 \subseteq N'$.

Recall that in a Polish space there is no family $\{K_\alpha,\ \alpha \in \omega_1\}$ of closed subset, indexed by the first uncountable ordinal ω_1, with $K_\alpha \subsetneq K_\beta$ for all $\beta < \alpha < \omega_1$.

Therefore, there exists $M \subseteq N'$ so that

$$K(M') = K(M) \quad \text{for all } M' \subseteq_a M.$$

Indeed, otherwise by a diagonalization argument we could construct $(K(M_\alpha))_{\alpha<\omega_1}$ so that $K(M_\alpha) \subsetneq K(M_\beta)$ for all $\beta < \alpha < \omega_1$.

Claim $\forall M' \subseteq_a M$, for all open $U \subseteq K(M)$, there are $M'' \subseteq_a M'$, $y, z \in U$ such that

$$(3) \qquad \lim_{n\in M''} f_n(y) \geq r + \delta$$

and

$$\lim_{n\in M''} f_n(z) \leq r.$$

Indeed, fix $M' \subseteq M$. Then $K(M') = K(M)$. By definition, there exists $y \in U : f_n(y) > r + \delta$ for infinitely many $n \in M'$. Now choose a subset $M_1 \subseteq_a M'$ such that $(f_n(y))_{n\in M_1}$ converges.

By definition, there exists $z \in U : f_n(z) < r$ for infinitely many $n \in M_1$. Finally, choose $M_2 \subseteq_a M_1$ so that $(f_n(z))_{n\in M_2}$ converges.

Now, let $(U_n)_n$ be a base of open sets of $K(M)$. Therefore, we can have $(M_n)_n$ a sequence of infinite sets of ω with

$M_{n+1} \subseteq_a M_n$ for all $n \in \omega$,

$z_n, y_n \in U_n$ for all $n \in \omega$,

such that the (3) of the claim holds.

As always, by diagonalization argument, let us consider $Q \subseteq_a M_n\ \forall n \in \omega$ and let $L = \{y_n, z_n :\ n \in \omega\}$. Notice that L is dense in $K(M)$.

Let us define

$$f(x) = \lim_{n\in Q} f_n(x) \quad \forall x \in L.$$

Consequently, $(f_{n_k})_k = (f_n)_{n\in Q}$, L and f satisfy the conclusion of the theorem. □

Theorem 1.22. *(H. Rosenthal)*
Let X be a Polish space and let F be a subset of $\mathcal{B}_1(X)$. The following are equivalents

(1) F is relatively compact;

(2) F is relatively countably compact;

(3) F is relatively sequentially compact.

Moreover, suppose F satisfies the equivalence, then

(a) every function in the closure of F is in the closure of a countable subset of F;

(b) if F is uniformly bounded and $(f_\alpha)_\alpha$ is a convergent net of F with limit f, then

$\int f_\alpha \, d\mu \longrightarrow \int f \, d\mu$ *for all signed Borel measure μ on X.*

Proof. (2) $\Rightarrow$ (3) By hypothesis, F has to be pointwise bounded. Then (3) holds by the previous theorem.

(2) $\Rightarrow$ (1) Suppose (1) fails. For (2), F is pointwise bounded; hence the pointwise closure of F in $X^{\mathbb{R}}$ is compact by Tychonoff's theorem. Therefore, there must exists a non 1th Baire class function f in the pointwise closure of F. By Baire's theorem 1.9, there exists a closed non empty subset K of X such that $f|_K$ has no point of continuity relative to K.

Claim: f satisfies the Discontinuity Criterion.

Indeed, for each $n \in \omega$ let

$$A_n = \{x \in K : \text{ for every neighborhood } U \text{ of } x \ \exists y, z \in U : f(y) - f(z) > \frac{1}{n}\}$$

Since $f|_K$ has no point of continuity, we have that

$$K = \bigcup_{n \in \omega} A_n.$$

By the Baire category's theorem 1.4, there is a n_0 such that A_{n_0} has non empty interior U_0. Let $K_0 = \overline{U_0}$ and $\delta = \frac{1}{n_0}$. We have that, for all $U \subseteq K_0$ open, $U \cap U_0$ is open in K_0. Then $\exists y, z \in U : f(y) - f(z) > \delta$.

Let $(r_n)_n = \mathbb{Q}$ and for $n \in \omega$ let us define

$$B_n = \{x \in K_0 : \text{ for every neighborhood } U \text{ of } x \ \exists y, z \in U \cap K_0 : \\ f(z) < r_n \\ f(y) > r_n + \delta\}$$

Then
$$K_0 = \bigcup_{n\in\omega} B_n.$$

Again, by the Baire category's theorem 1.4, $\exists n_1 \in \omega$ such that B_{n_1} has non empty interior V. Let us consider $L = \overline{V}$ and $r = r_{n_1}$.Then, we have that f satisfies the Discontinuity Criterion for L, r, δ.

Let $(U_n)_n$ be a base of open sets in L. For each $n \in \omega$ choose $y_n, z_n \in U_n$ such that
$$f(y_n) > r + \delta \qquad f(z_n) < r.$$
Let $Q = \{y_n, z_n : \ n \in \omega\}$. Since f is in the pointwise closure of F and Q is a countable set, there must exists a sequence $(f_n)_n \subseteq F$ such that
$$f_n(q) \overset{n\to\infty}{\longrightarrow} f(q) \qquad \forall q \in Q$$
But Q is dense in L, it follows that $f|_Q$ satisfies the Discontinuity Criterion. Moreover, it is clear that if g is a cluster point of $(f_n)_n$ then $g|_Q = f|_Q$. Therefore, g has no point of continuity in $\overline{Q}$. Thus $(f_n)_n$ has no 1th Baire class cluster point. That means (2) fails.

Since (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (2) are trivial, we have that the equivalence of (1) $-$ (2) $-$ (3).

To show (2) $\Rightarrow$ (a) we need the following □

Lemma 1.23. *Let S be a pointwise relatively compact of $\mathcal{B}_1(X)$, $0 \in \overline{S}$, $s(x) \geq 0$ for all $s \in S$, $x \in X$.*

Then, $\forall \delta > 0$ $\exists H \subseteq S$ a countable set such that
$$\inf_{h\in H} h(x) < \delta \qquad \forall x \in X.$$

Proof. Suppose not. Then $\forall H \subseteq S$ $\exists \delta > 0$ such that
$$K(H) = \{x \in X : \ h(x) \geq \delta \ \ \forall h \in H\}$$
is non empty. Then we have
$$K(H_1) \subseteq K(H_2) \quad \text{whenever} \quad H_2 \subseteq H_1.$$
By transfinite induction, we construct $(D_\alpha)_{\alpha<\omega_1}$, $((s_n^\alpha)_{n\in\omega})_{\alpha<\omega_1} \subseteq S$ and $(H_\alpha)_{\alpha<\omega_1}$ so that

(i) $H_\alpha \subseteq H_\beta$ for $\alpha < \beta < \omega_1$;

(ii) D_α is dense in $\overline{K(H_\alpha)}$ and D_α countable;

(iii) $\lim_n s_n^\alpha(x) = 0$ for all $x \in D_\alpha$;

(iv) $H_{\alpha+1} = H_\alpha \cup \{s_n^\alpha, \quad n \in \omega\}$.

Let H_0 be arbitrary. Chosen H_α and D_α, we can consider $((s_n^\alpha)_{n\in\omega})_{\alpha<\omega_1} \subseteq S$ as in (iii) by a diagonalization argument and using the fact that $0 \in \overline{S}$.

Let us consider $H_{\alpha+1}$ as in (iv). If β is a limit ordinal, put $H_\beta = \bigcup_{\alpha<\beta} H_\alpha$. the countability of β and the countability of every H_α insures that H_β is countable.

Then there must exists $\alpha < \omega_1$ such that $K(H_\alpha) = K(H_{\alpha+1})$.

Let f be any cluster point of $((s_n^\alpha)_{n\in\omega})_{\alpha<\omega_1}$. Then f must vanish on D_α.

$$\forall x \in K(H_{\alpha+1}),\ s_n^\alpha(x) \geq \delta \text{ for all } n \in \omega \ \Rightarrow\ f(x) \geq \delta.$$

Since $K(H_{\alpha+1})$ and D_α are dense in $\overline{K(H_\alpha)}$ we have

f satisfies the Discontinuity Criterion

$\Rightarrow\ f \notin \mathcal{B}_1(X)$. A contradiction. □

Proof. of (2) $\Rightarrow$ (a)

$\forall m \in \omega$ let

$$\phi_m : \mathcal{B}_1(X) \longrightarrow \mathcal{B}_1(X^m)$$

define by

$$\phi_m(f)(x_1, \dots x_m) = |f(x_1)| + \dots + |f(x_m)|.$$

Let $g \in \overline{F}$. WLOG we can suppose $g = 0$ (otherwise consider $\{f - g :\ f \in F\}$). Therefore, ϕ_m is a continuous map and $\phi_m(0) = 0$. Then $\phi_m(F)$ is relatively compact of $\mathcal{B}_1(X^m)$ and $0 \in \overline{\phi_m(F)}$. By Lemma 1.23, there must exists H_m a countable set of F such that

$$\frac{1}{m} > \inf\{(\phi_m h)(y),\ h \in H_m\} \qquad \forall y \in X^m$$

$$\Rightarrow 0 \in \overline{\bigcup_{m\in\omega} H_m}.$$

□

To show (1) $\Rightarrow$ (b) we shall need of the following

Lemma 1.24. *Let X be a compact Hausdorff space and let us denote by K the unit ball of $M(X)$ (the space of all bounded signed Borel regular measures on X) endowed with the weak* topology relative to $C(X)$.*

Let us define

$$T : bd - \mathcal{B}_1(X) \longrightarrow K^{\mathbb{R}}$$

by

$$Tf(\mu) = \int_X f \ d\mu,$$

where we are denoting by $bd - \mathcal{B}_1(X)$ the space of all 1th Baire class which are bounded.

Then the range of T is a closed subset of $bd - \mathcal{B}_1(K)$.

Proof. It enough to show that $T(bd - \mathcal{B}_1(X))$ consists of all functions in $bd - \mathcal{B}_1(K)$ which are antisymmetric and affine.

Obvious all functions in $T(bd - \mathcal{B}_1(X))$ are antisymmetric and affine. Let us suppose $f \in \mathcal{B}_1(K)$ bounded, antisymmetric and affine. Then there exists an element $\widetilde{f} \in M(X)^* = C(X)^{**}$ such that

$$\widetilde{f}|_K = f.$$

Claim: $\widetilde{f}$ is of 1th Baire class $\Longleftrightarrow \widetilde{f}|_K = f$ is of 1th Baire class.

Suppose we have already proved the Claim, then $\widetilde{f}|_K = f$ is of 1th Baire class. Therefore, $\widetilde{f} \in M(X)^*$ is of 1th Baire class.

Then there exists $(f_n)_n \subseteq C(K)$ such that

$$\lim_n \langle \mu, f_n \rangle = \langle \mu, \widetilde{f} \rangle \quad \forall \mu \in M(X).$$

But

$$\langle \mu, f_n \rangle = \int f_n \ d\mu \quad \forall \mu \in K$$

By the Lebesgue convergent's theorem

$$\exists h \in \mathcal{B}_1(X) \ : \ \langle f, \mu \rangle = \int h \ d\mu,$$

or

$$f = T(h).$$

Let us prove the Claim above.

Actually the Claim holds in a more general setting.

Let X be a Banach space, $K = (B_{X^*}, weak^*)$, $f \in X^{**}$. Then

$$f \text{ is of 1th Baire class } \iff f|_K \text{ is of 1th Baire class.}$$

Subcalim: Let X be a subspace of Y and $G \in X^{**} \subseteq Y^{**}$. If G is of 1th Baire class in Y^{**} then G is of 1th Baire class in X^{**}.

Indeed, assuming $\|G\| = 1$. If there exists $(b_n)_n \subseteq Y$ such that $b_n \overset{n\to\infty}{\longrightarrow} G$ weak* (or pointwise). We show that

$$d(B_X, \overline{co}\{b_N, b_{N+1}, \ldots\}) = 0, \quad \forall N \in \omega,$$

or it is the same to say that we can choose $(x_n)_n \subseteq X$ and $\overline{b_n}$ convex combination of b_n's such that

$$\|x_n - \overline{b_n}\| \longrightarrow 0.$$

Indeed, since $\overline{b_n} \longrightarrow G$ weakly* (on Y^*), then

$$x_n \longrightarrow G \text{ weakly}^* \text{ (on } Y^*) \text{ and for Hahn-Banach}$$

$$x_n \longrightarrow G \text{ weakly}^* \text{ (on } X^*).$$

If there exists $N \in \omega$ such that $d(B_X, \overline{co}\{b_N, b_{N+1}, \ldots\}) > 0$, by the Hahn-Banach separation

$$\exists f \in Y^* \ : \ \sup_{x \in B_X} f(x) < \inf_{j \geq N} f(b_j).$$

By Goldstein's theorem

$$|G(f)| \leq \sup_{x \in B_X} |f(x)| < \inf_{j \geq N} f(b_j) \leq \lim_{j \to \infty} f(b_j) = G(f)$$

Now, suppose $f \in C(X)^{**}$ is such that $f|_K$ is of 1th Baire class.

Let us denote by $supp\mu = \{x \in X : \ |\mu|(U) > 0 \, \forall U \text{ open neighborhood of } x\}$ with $\mu \in M(X)$. For $S \subseteq X$ let us denote by

$$\mathcal{P}(S) = \{\mu \in M(X) : \ \|\mu\| = 1, \ supp\mu \subseteq S\}.$$

Then, $\mathcal{P}(S)$ is a weak* closed of K.

Suppose f is not of 1th Baire class on $(C(X)^*, weak^*)$. We want to show that $\exists \mu \in M(X)$ such that

$$f|_{\mathcal{P}(supp\mu)} \text{ has no point of continuity in } \mathcal{P}(supp\mu).$$

Let us consider

$\mathcal{P}_d(S)$ the set of all purely atomic member of $\mathcal{P}(S)$. Notice that it is weak* dense in $\mathcal{P}(S)$;

$\mathcal{P}_\mu(S)$ the set of all μ-continuous members of $\mathcal{P}(S)$.

If either $Y = \mathcal{P}_d(S)$ or $Y = \mathcal{P}_\mu(S)$ then Y is convex and

$$\|f\|_\infty = \sup_{\nu \in Y} |\int f \; d\mu| \quad \forall f \in C(S).$$

Obvious $X \hookrightarrow K = B_{M(X)}$, then $f|_X$ is of 1th Baire class.

Let us define $g \in C(X)^{**}$ by

$$g(\mu) = \int f(\xi) \; d\mu(\xi), \quad \forall \mu \in M(X).$$

Of course, $g \in \mathcal{B}_1(C(X)^*)$. Then,

$$h = f - g \in \mathcal{B}_1(K).$$

Let us show that $h = 0$.

By definition of h we have that $h(\mu) = 0$ for all $\mu \in \mathcal{P}_d(X)$.

If $h \neq 0$, then $\exists \nu \in \mathcal{P}(X) : h(\nu) \neq 0$ (we can suppose $h(\nu) > 0$). By the Radon-Nikodym's theorem, we have that

$$\mathcal{P}_\nu(X) = L_1(\nu)$$

Then

$$h|_{\mathcal{P}_\nu(X)} \text{ is a bounded linear functional on } \mathcal{P}_\nu(X).$$

By Riesz representation's theorem, there exists a bounded Borel measurable function ϕ such that

$$h(\lambda) = \int \phi \; d\lambda \qquad \forall \lambda \in \mathcal{P}_\nu(X)$$

In particular $h(\nu) = \int \phi \; d\nu > 0$. Which implies

$$\int \phi^+ \; d\nu > 0$$

Let $c > 0$ such that $\nu(E) > 0$ where $E = \{\xi : \; \phi(\xi) \geq c\}$. It follows that

$$\text{if } \lambda \in \mathcal{P}(X) : \; \lambda(X \setminus E) = 0 \Rightarrow \int \phi \; d\lambda = \int_E \phi \; d\lambda \geq c.$$

Let us define $\mu \in \mathcal{P}(X)$ as

$$\mu(B) = \frac{\nu(B \cap E)}{\nu(E)}.$$

Then $h(\lambda) \geq c$ for all $\lambda \in \mathcal{P}_\mu(X)$.

Let $S = supp\mu$. Then

$$h \geq c \text{ on } \mathcal{P}_\mu(X) \text{ (which is weak}^*\text{-dense in } \mathcal{P}(X)\text{), and}$$

$$h = 0 \text{ on } \mathcal{P}_d(X) \text{ (which is weak}^*\text{-dense in } \mathcal{P}(X)\text{)}$$

$\Rightarrow$ $h|_{\mathcal{P}(S)}$ has no point of continuity in $\mathcal{P}(S)$. But $\mathcal{P}(S) \subseteq K$ and $h|_K \in \mathcal{B}_1(K)$. Namely a contradiction. □

Proof. of $(1) \Rightarrow (b)$

$F \subseteq \mathcal{B}_1(X)$ is relatively compact.

If X is compact, by Lemma 1.24, $T(F) \subseteq bd{-}\mathcal{B}_1(X)$ is relatively compact.

If X is not compact, let $(f_\alpha) \subseteq F$ be a net such that $f_\alpha \longrightarrow f$, $c = \sup_\alpha |f_\alpha|$ and $\mu \in M(X)$.

By Ulam's theorem, given $\varepsilon > 0$ $\exists K \subseteq X$ compact : $|\mu|(X \setminus K) < \varepsilon$.

Therefore, the restriction map $\mathcal{B}_1(X) \longrightarrow \mathcal{B}_1(K)$ is continuous (easy!). Then $F|_K$ is relatively compact in $\mathcal{B}_1(K)$. By all considerations above

$$\int_K f_\alpha \, d\mu \longrightarrow \int_K f \, d\mu.$$

Consequently,

$$\limsup_\alpha |\int (f_\alpha - f) \, d\mu| \leq \limsup_\alpha \int_{X \setminus K} |f_\alpha - f| \, d\mu \leq 2c\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have (b). □

Definition 1.25. A topological space (X, θ) is called *Cech-complete* if it can be considered as a G_δ subset of a compact Hausdorff space; i.e., there exists a compact Hausdorff space Z and a countable family of open $(A_n)_n$ in Z so that $X = \bigcap_n A_n$.

Remark 1.26. (i) Any locally compact Hausdorff space is Chec-complete (being open in its one-point compactification);

(ii) any complete metric space is Cech-complete (being G_δ in its Cech-Stone compactification).

Before to enunciate the main result of this section, we shall need a bunch of lemmas

Lemma 1.27. *Let X be a Cech-complete space and $\mathcal{A}$ a family of pairs (A, B), with $A, B \subseteq X$ are open's.*

Suppose there is $Y \subseteq X$ non empty so that $\underline{\mathcal{A} \text{ is weakly dense over } Y}$ (i.e., $\forall E_0, \ldots, E_n \subseteq X$ open's : $E_k \cap Y \neq \emptyset$, $k = 0, \ldots, n$, then $\exists (G, H) \in \mathcal{A}$ such that $G \cap E_i \cap Y \neq \emptyset$, $H \cap E_i \cap Y \neq \emptyset$ for all $i = 0, \ldots, n$).

Then there is $(G_n, H_n)_n \subseteq \mathcal{A}$ and a compact set $K \subseteq X$ such that

$$K \cap \bigcap_{n \in I} G_n \cap \bigcap_{n \in \omega \setminus I} H_n \neq \emptyset \quad \forall I \subseteq \omega.$$

Proof. By hypothesis, there is a compact Hausdorff space Z and $(A_n)_n$ open subsets in Z such that $X = \bigcap_{n \in \omega} A_n$.

Let

$$\mathcal{B} = \{(G, H) : \ G, H \subseteq Z \text{ open's}, \ (G \cap X, H \cap X) \in \mathcal{A}\}.$$

We have that $\mathcal{B}$ is weakly dense over Y.

Claim: There exist $\{(G_n, H_n) : \ n \in \omega\}$ and open sets $C_{P,Q}$ in Z such that

(i) $C_{P,Q}$ is defined for pairs (P, Q) which is a partition of $\{0, \ldots, n\}$, for some $n \in \omega$, and $C_{P,Q}$ is a non empty open set in Z such that

$C_{P,Q} \cap Y \neq \emptyset$ and

$\overline{C_{P,Q}} \subseteq A_n \cap \bigcap_{n \in P} G_n \cap \bigcap_{n \in Q} H_n$.

(ii) If $P \subseteq P'$ and $Q \subseteq Q'$, then $C_{P',Q'} \subseteq C_{P,Q}$.

As Y is non empty, by hypothesis there is $(G_0, H_0) \in \mathcal{B}$ such that

$$G_0 \cap Y \neq \emptyset \qquad H_0 \cap Y \neq \emptyset.$$

Choose a non empty open sets $C_{\{0\},\emptyset}, C_{\emptyset,\{0\}}$ in Z such that

$$C_{\{0\},\emptyset} \cap Y \neq \emptyset \qquad C_{\emptyset,\{0\}} \cap Y \neq \emptyset.$$

and

$$\overline{C_{\{0\},\emptyset}} \subseteq G_0 \cap A_0, \qquad \overline{C_{\emptyset,\{0\}}} \subseteq H_0 \cap A_0.$$

Suppose that G_i, H_i have been chosen for all $i \leq n$ and $C_{P,Q}$ has been found for each partition (P, Q) of $\{0, \ldots, n\}$.

Each $C_{P,Q}$ is a non empty open set in Z such that $C_{P,Q} \cap Y \neq \emptyset$. As $\mathcal{B}$ is weakly dense over Y, $\exists (G_{n+1}, H_{n+1}) \in \mathcal{B}$:

$$G_{n+1} \cap C_{P,Q} \cap Y \neq \emptyset, \qquad H_{n+1} \cap C_{P,Q} \cap Y \neq \emptyset,$$

for every partition (P,Q) of $\{0,\ldots,n\}$. Now, for every partition (P,Q) of $\{0,\ldots,n\}$ choose $C_{P\cup\{n+1\},Q}$ and $C_{P,Q\cup\{n+1\}}$ two open sets such that

$$C_{P\cup\{n+1\},Q} \cap Y \neq \emptyset, \qquad C_{P,Q\cup\{n+1\}} \cap Y \neq \emptyset$$

and

$$\overline{C_{P\cup\{n+1\},Q}} \subseteq G_{n+1} \cap A_{n+1}, \qquad \overline{C_{P,Q\cup\{n+1\}}} \subseteq H_{n+1} \cap A_{n+1}.$$

Let us define

$$K = \bigcap_{n\in\omega} \bigcup \{\overline{C_{P,Q}} : \ (P,Q) \text{ is a partition of } \{0,\ldots,n\}\}.$$

Then K is closed in Z and then compact. For $I \subseteq \omega$, let

$$P_n = \{i \in I : \ i \leq n\} \text{ and } Q_n = \{i \notin I : \ i \leq n\},$$

(P_n, Q_n) is a partition of $\{0,\ldots,n\}$. Since Z is compact

$$\emptyset \neq \bigcap_{n\in\omega} \overline{C_{P_n,Q_n}} \subseteq K \cap \bigcap_{n\in I} G_n \cap \bigcap_{n\in\omega\setminus I} H_n.$$

Finally, as $\overline{C_{P,Q}} \subseteq A_n$ for each partition (P,Q) of $\{0,\ldots,n\}$, we get

$$K \subseteq X.$$

□

Lemma 1.28. *Let X be a regular Hausdorff space which is sequentially compact and such that*

(*C*) *if $A \subseteq X$, $x \in \overline{A}$, there exists a countable set $A_0 \subseteq A$: $x \in \overline{A_0}$.*

Let $(x_n)_n$ be a sequence in X and $(I_n)_n$ be a decreasing sequence of infinite subsets of ω such that

$$(x_i)_{i\in I_n} \text{ have a common cluster point } x.$$

Then there is an infinite set $I \subseteq \omega$: $I \setminus I_n$ is finite, for all $n \in \omega$, and x is a cluster point of $(x_i)_{i\in I}$.

Proof. Let

$$F = \{\lim_{i\in I} x_i : \ I \text{ is an infinite set, } \lim_{i\in I} x_i \text{ exists and } I \setminus I_n \text{ is finite } \forall n \in \omega\}.$$

Claim: $x \in \overline{F}$.

For a neighborhood U of x, let $J = \{i \in \omega : \ x_i \in U\}$. Then $J \cap I_n$ is a infinite set.

As $(I_n)_n$ is decreasing, there is an infinite $K \subseteq J$: $K \setminus I_n$ is finite $\forall n \in \omega$. Now, X is sequentially compact. Therefore, there is an infinite $I \subseteq K$ such that

$$z = \lim_{i\in I} x_i$$

exists.

We have $z \in F \cap \overline{U}$. Since X is regular, $x \in \overline{F}$.

By hypothesis (C), there is $(z_m)_m \subseteq F$ such that $x \in \overline{\{z_m : \ m \in \omega\}}$. Every

$$z_m = \lim_{i\in J_m} x_i$$

where J_m is infinite: $J_m \setminus I_n$ is finite $\forall n \in \omega$.

Let $I = \bigcup_{n\in\omega}(I_n \cap J_n)$. Then

$$I \setminus I_n \text{ is finite, and } J_n \setminus I \text{ is finite, } \forall n \in \omega.$$

Follows that z_m is a cluster point of $(x_i)_i$. But the set of cluster points of a sequence is always closed. Thus, x is a cluster point of $(x_i)_{i\in I}$. □

Lemma 1.29. *Let X be a Polish space, $(x_n)_n$ a sequence in $C_p(X)$:*

(i) $\{x_n : \ n \in \omega\}$ is relatively compact in $\mathcal{B}_1(X)$;

(ii) 0 is a cluster point of $(x_n)_n$ in the pointwise topology.

Let $W \subseteq X$ be a non empty closed set and $\varepsilon > 0$. Then there is a non empty relatively open $U \subseteq W$ and an infinite $J \subseteq \omega$:

(a) 0 is a cluster point of $(x_i)_{i\in J}$;

(b) $\limsup_{i\in J} |x_i(t)| \leq 2\varepsilon$ for all $t \in U$.

Proof. $\forall I \subseteq \omega$ infinite, let

$$A(I) = \{\text{cluster points of } (x_i)_{i\in I}\} \subseteq \mathcal{B}_1(X).$$

Suppose the Lemma fails. If

$$G_i = \{t \in X : \ |x_i(t)| < \varepsilon\}, \quad H_i = \{t \in X : \ |x_i(t)| > 2\varepsilon\},$$

let

$$\mathcal{A} = \{(G_i, H_i) : \ i \in \omega\}.$$

Claim: $\mathcal{A}$ is weakly dense over W.

Indeed, let $E_0, \dots, E_n \subseteq X$ open sets with $E_i \cap W \neq \emptyset$, $i = 0, \dots, n$.

Let $s_i \in E_i \cap W$, $i = 0, \dots, n$ and

$$I = \{i \in \omega : \ |x_i(s_r)| < \varepsilon, \ \forall r \leq n\}.$$

Then, by (ii) above, $0 \in A(I)$. Let

$$J_r = \{i \in I : \ |x_i(t)| \leq 2\varepsilon, \ \forall t \in E_r \cap W\}.$$

By our hypothesis, $0 \notin A(J_r)$ for any $r \leq n$. Since

$$A(\bigcup_{r \leq n} J_r) = \bigcup_{i \leq n} A(J_r),$$

it follows that $I \neq \bigcup_{n \geq r} J_r$. If i is any point of $I \setminus \bigcup_{r \leq n} J_r$, we have

$$G_i \cap E_r \cap W \neq \emptyset \text{ (as } i \in I) \quad H_i \cap E_r \cap W \neq \emptyset \text{ (as } i \notin J_r).$$

By Lemma 1.27, there exists $K \subseteq X$ compact such that

$$K \cap \bigcap_{n \in I} G_n \cap \bigcap_{n \in \omega \setminus I} H_n \neq \emptyset, \quad \forall I \subseteq \omega.$$

In particular, there is a sequence $(y_n)_n$ in $\{x_i, \ i \in \omega\}$ such that, for every $I \subseteq \omega$

$$\{t \in K : \ |y_n(t)| < \varepsilon, \ \forall n \in I, \ |y_n(t)| > 2\varepsilon \ \forall n \in \omega \setminus I\} \neq \emptyset.$$

It follows that $(|y_n|)_n$ can have no convergent subsequence (as well as $(y_n)_n$). But $(y_n)_n$ is a sequence in $\{x_i : \ i \in \omega\}$ which is relatively compact. By Theorem 1.22, it is relatively sequentially compact in $\mathcal{B}_1(X)$. A contradiction. $\square$

Lemma 1.30. *Let X be a Polish space, $(x_n)_n$ be a sequence in $C_p(X)$ such that*

(i) $\{x_n : \ n \in \omega\}$ is relatively compact in $\mathcal{B}_1(X)$;

(ii) 0 *is a cluster point of* $(x_n)_n$.

Then there is an infinite set $I \subseteq \omega$ *such that*

$$\limsup_{i\in I} |x_i(t)| \leq \varepsilon, \quad \forall t \in X$$

and 0 *is a cluster point of* $(x_i)_{i\in I}$.

Proof. For each $I \subseteq \omega$, let

$$U(I) = int\{t : \limsup_{i\in I} |x_i(t)| \leq \varepsilon\}$$

and $A(I)$ the set of all cluster points of $(x_i)_{i\in I}$.

Note that, if $I \setminus J$ is finite $\Rightarrow U(I) \supseteq U(J)$.

Let $(V_k)_k$ be a base of X and let us start with $I_0 = \omega$. Given I_k such that $0 \in A(I_k)$. Then, if there is an infinite $I \subseteq I_k$: $0 \in A(I)$ and $V_k \subseteq U(I)$, take $I_{k+1} = I$. Otherwise choose $I_{k+1} = I_k$.

Therefore, the sequence $(I_k)_k$ is decreasing: $0 \in A(I_k)$ for all $k \in \omega$.

By Lemma 1.28 for the set $\overline{\{x_i,\ i \in \omega\}}$ there is an infinite $I \subseteq \omega$ such that

$$0 \in A(I) \text{ and } I \setminus I_k \text{ is finite } \forall k \in \omega.$$

Fix $J \subseteq I$ infinite such that $0 \in A(J)$. Then, $U(J) \supseteq U(I)$.

If $U(J) \neq U(I)$, there should exists $k \in \omega$ such that $V_k \subseteq U(J)$ but $V_k \nsubseteq U(I)$.

Since $J \setminus I_k$ is finite, it follows that $J \cap I_k$ is infinite in I_k: $0 \in A(J \cap I_k)$ and $V_k \subseteq U(J \cap I_k)$ (for construction of I_k).

Therefore, $V_k \subseteq U(I_{k+1})$. But in this situation $I \setminus I_{k+1}$ has to be finite, so that

$$V_k \subseteq U(I_{k+1}) \subseteq U(I),$$

which contradicts the assumption above.

What we have is:

$$(a) \quad U(J) = U(I) \quad \forall J \subseteq I : \ 0 \in A(J).$$

Claim: $U(I) = X$.

Suppose not. Let $W \subseteq X \setminus U(I)$ be a non empty closed set. By the Lemma 1.29 applied to $(x_i)_{i\in I}$ there exists $J \subseteq I$: $0 \in A(J)$ and

$$\limsup_{i\in J} |x_i(t)| \leq \varepsilon, \ \forall t \in U, \text{ where } U \text{ is some open of } W.$$

Thus

$$\limsup_{i \in J} |x_i(t)| \leq \varepsilon, \ \forall t \in U \cup U(I)$$

and

$$U(J) \subseteq int[U \cup U(I)] \neq U(I).$$

Which contradicts (a) above. □

Corollary 1.31. *Let X be a Polish space, $(x_n)_n$ be a sequence in $C_p(X)$ such that*

(i) $(x_n)_n$ *is relatively compact;*

(ii) 0 *is a cluster point of* $(x_n)_n$.

Then, there is a subsequence of $(x_n)_n$ converging to 0.

Proof. By Lemma 1.30, for $\varepsilon = \frac{1}{2^k}$ $\exists I_k \subseteq \omega$, $k \in \omega$ so that

$$\limsup_{i \in I_k} |x_i(t)| \leq \frac{1}{2^k}, \quad \forall t \in X, \ k \in \omega.$$

Notice that we can always choose $(I_k)_k$ decreasing. Therefore, let us consider $I \subseteq \omega$: $I \setminus I_k$ is finite $\forall k \in \omega$. that implies

$$\lim_{i \in I} x_i = 0.$$

□

Here we are ready to enunciate the main result

Theorem 1.32. *(Bourgain-Fremlin-Talagrand)*
If X is a Polish space, then $\mathcal{B}_1(X)$ is angelic.

Proof. Actually, Theorem 1.22 says us that every relatively countably compact is relatively compact in $\mathcal{B}_1(X)$.

We need to show the other condition of angelicity.

Let us consider $A \subseteq \mathcal{B}_1(X)$ a relatively compact, $x \in \overline{A}$. By Theorem 1.22(a), there is a sequence $(x_n)_n \subseteq A$ such that x is a cluster point of $(x_n)_n$.

Let us define

$$\varphi : X \longrightarrow \mathbb{R}^\omega$$

given by

$$\varphi(t)(0) = x(t)$$

$$\varphi(t)(n+1) = x_n(t),$$

for all $t \in X$ and $n \in \omega$.

1. *φ is a Borel map.*

It is enough to show that, if $n_i, \dots n_k \in \omega$, then

$$\varphi^{-1}(\{f \in \mathbb{R}^\omega : \ |f(n_i)| < \sigma, \ i = 1, \dots, k\}$$

is Borel.

But this set coincides with

$$\{t \in X : \ |\varphi(t)(n_i)| < \sigma, \ i = 1, \dots, k\} = \bigcap_{i=1}^{k} \{t \in X : \ |x_{n_i - 1}| < \sigma\}.$$

Since each x_n set in $\mathcal{B}_1(X)$, we have that $\{t \in X : \ |x_{n_i-1}| < \sigma\}$ is a G_δ set (the inverse imageof an open set through a 1th Baire class function is a G_δ). Therefore, φ is Borel.

2. Let us consider $\{(x, y) : \ \varphi(x) = y\} \subseteq X \times \mathbb{R}^\omega$. Letting

$$h(x, y) = |y - \varphi(x)|$$

we have that h is a Borel map. Since

$$\{(x, y) : \ \varphi(x) = y\} = h^{-1}(0)$$

we have that $L = \{(x, y) : \ \varphi(x) = y\}$ is Borel in $X \times \mathbb{R}^\omega$.

Let us denote by

$$P : X \times \mathbb{R}^\omega \longrightarrow \mathbb{R}^\omega$$

the second projection. Since $Y = \varphi(X)$ coincides with $P(L)$, we have that Y is an analytic set. From what we have seen in the Tertulia seminar [5], there must exists a polish space Z and a continuous surjection

$$\psi : Z \longrightarrow Y = \phi(X) \subseteq \mathbb{R}^\omega.$$

Set

$$y(u) = u(0)$$
$$y_n(u) = u(n + 1),$$

as elements of $\mathbb{R}^Y$.

We see that y is a cluster point of $(y_n)_n$ in $\mathbb{R}^Y$ and every subsequence of $(y_n)_n$ has a convergent subsequence (this because A is relatively sequentially compact in $\mathcal{B}_1(X)$), then every subsequence of $(x_n)_n$ has a convergent subsequence).

Let

$$z = y \circ \psi$$
$$z_n = y_n \circ \psi$$

as element of $\mathbb{R}^Z$. Therefore, z is a cluster point of $(z_n)_n$ in $\mathbb{R}^Z$. Notice that $(z_n)_n \subseteq C_p(Z)$ (since each y_n is continuous projection coordinate and ψ is continuous). Moreover every subsequence of $(z_n)_n$ has a convergent extract. By Rosenthal's theorem (Theorem 1.22), $\{z_n,\ n \in \omega\}$ is relatively compact in $\mathcal{B}_1(Z)$. We also have $z \in C_p(Z)$. Let us apply Corollary 1.31 to $(z_n - z)_n$ to get a subsequence $(z_n)_{n\in I}$ convergent to z. By construction, first we have

$$\lim_{n\in I} y_n = y$$

and secondly

$$\lim_{n\in I} x_n = x$$

as required. □

Let us give another characterization of 1th Baire class function in the same spirit of the Baire characterization theorem.

Lemma 1.33. *(Talagrand)*
Let X be a complete metric space, $x \in \mathbb{R}^X$. Then $x \in \mathcal{B}_1(X)$ if and only if $x|_K \in \mathcal{B}_1(K)$, for every compact $K \subseteq X$.

Proof. Of course, one way of it is trivial. All we need to show is that if $x|_K \in \mathcal{B}_1(X)$ for every compact $K \subseteq X$, then $x \in \mathcal{B}_1(X)$.

By Baire characterization theorem (see Theorem 1.9), it is enough to show that for every closed $M \subseteq X$, $f|_M$ has a point of continuity relative to M. For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, let us denote

$$S(\alpha) = \{t \in M :\ x(t) \leq \alpha\}, \qquad T(\beta) = \{t \in M :\ x(t) \geq \beta\}.$$

Therefore, it is enough to show that whenever $\alpha < \beta$,

$$int\overline{S(\alpha)} \cap int\overline{T(\beta)} = \emptyset,$$

or it is the same to say that: $x \in \mathcal{B}_1(X)$ if and only if for every closed closed set $M \subseteq X$ and $\alpha < \beta$,

$$\text{one of } \overline{M \cap S(\alpha)},\quad \overline{M \cap T(\beta)} \text{ is not equal to } M.$$

Suppose $x \notin \mathcal{B}_1(X)$. Then there is a non empty closed set $M \subseteq X$, $\alpha < \beta$ reals, so that

$$S(\alpha),\ T(\beta) \text{ are dense in } M$$

By induction, we can choose a sequence of finite sets $A_n \subseteq S(\alpha) \cup T(\beta)$ such that

(i) $A_0 \neq \emptyset$;

(ii) A_n A_{n+1}, $\forall n \in \omega$;

(iii) $\forall t \in A_{n+1}$, $\exists s \in A_n$ such that $d(t,s) \leq \frac{1}{2^n}$;

(iv) $\forall s \in A_n$, $\exists t \in A_{n+1}$ such that $d(t,s) \leq \frac{1}{2^n}$ and $|x(s) - x(t)| \geq \beta - \alpha$.

Since X is a complete metric space, then

$$K = \overline{\bigcup_{n \in \omega} A_n}$$

is compact (because it is complete and totally bounded).

Hence $K \cap S(\alpha)$, $K \cap T(\beta)$ are dense in K. That implies $x|_K \notin \mathcal{B}_1(K)$. A contradiction. □

Lemma 1.34. *(Talagrand)*
Let X be a complete metric space. Then

$x \in \mathcal{B}_1(X)$ if and only if for every non empty open $U \subseteq X$, $\varepsilon > 0$ there is a non empty open $V \subseteq U$ such that $diam(x(V)) \leq \varepsilon$.

Proof. If $x \in \mathcal{B}_1(X)$, we already know that the points of continuity of x is dense in X. Therefore, if we fix $t \in U$ and consider the continuity condition, we have that the condition is trivially satisfies.

Suppose we start with the condition, but $x \notin \mathcal{B}_1(X)$.

Thus, we can consider $E, F \subseteq \mathbb{R}$ closed and disjoint such that

$$\text{if } U = int\overline{x^{-1}(E)} \cap int\overline{x^{-1}(F)} \neq \emptyset$$

then $U \cap x^{-1}(E)$ and $U \cap x^{-1}(F)$ are dense in U.

By our condition, we can choose a sequence $(V_n)_n$ of non empty open sets in X such that

(i) $V_0 \subseteq U$;

(ii) $V_n \subseteq V_{n-1}$, for all $n \in \omega$;

(iii) $diam(x(V_n)) \leq \varepsilon$, for all $n \in \omega$;

Now,

$$V_n \cap x^{-1}(E) \neq \emptyset, \qquad V_n \cap x^{-1}(F) \neq \emptyset.$$

Let $s_n \in V_n \cap x^{-1}(E)$ and $t_n \in V_n \cap x^{-1}(F)$ for all $n \in \omega$. Therefore, $(x(s_n))_n$ and $(x(t_n))_n$ are two Cauchy sequences in $\mathbb{R}$ which must have a common limit in $E \cap F$ (which contradicts that E and F are disjoints). □

Let us fix some notation.

For any sets A, X and $S \subseteq A \times X$, let

$$\pi_1(S) = \{x \in A : \exists t \in X \ (x,t) \in S\},$$

$$S(x) = \{t \in X : (x,t) \in S\},$$
$$S^{-1}(t) = \{x \in A : (x,t) \in S\}.$$

Let Σ and $\mathcal{B}$ two σ-algebras of subsets of A and X respectively. $\Sigma \overset{\wedge}{\otimes} \mathcal{B}$ will denote the σ-algebra generated by $\{E \times F : E \in \Sigma, F \in \mathcal{B}\}$.

Lemma 1.35. *Let (A, Σ, μ) be a complete probability space and X be a compact metric space. Let $\mathcal{B}$ be the σ-algebra of Borel sets in X. Then*

$$\pi_1(S) \in \Sigma, \quad \forall S \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}.$$

Proof. Of course, we can write S in the form

$$S = \cup\{\cap_n E_{\phi|_n} \times F_{\phi|_n} : \phi \in \omega^\omega\},$$

where $\phi|_n = (\phi(0), \ldots, \phi(n))$, $E_{\phi|_n} \in \Sigma$ and $F_{\phi|_n}$ are closed in X.

Without loss in generality, we can assume, as well as we do, that

$$E_{\phi|_{n+1}} \subseteq E_{\phi|_n}, \quad F_{\phi|_{n+1}} \subseteq F_{\phi|_n}, \ \forall n \in \omega.$$

Therefore,

$$\begin{aligned}\pi_1(S) &= \cup\{\pi_1(\cap_n E_{\phi|_n} \times F_{\phi|_n}) : \phi \in \omega^\omega\} \\ &= \cup\{\cap_n \pi_1(E_{\phi|_n} \times F_{\phi|_n}) : \phi \in \omega^\omega\} \\ &= \cup\{\cap_n E_{\phi|_n} : \phi \in \omega^\omega\},\end{aligned}$$

which lies in Σ. □

Lemma 1.36. *Let (A, Σ, μ) be a complete probability measure and (X, d) be a compact metric space.*

Let S and T subsets of $A \times X$ such that

(*) $S^{-1}(t), T^{-1}(t) \in \Sigma$, *for all $t \in X$;*

(**) *for every $x \in A$ and every non empty closed $F \subseteq X$ at least one of the sets*

$\overline{F \cap S(x)}$, $\overline{F \cap T(x)}$ *is not equal to F.*

Then, for any $\delta > 0$ and any non epty open $U \subseteq X$ there is a non empty open $V \subseteq U$ such that

$$\mu(S^{-1}(s)) + \mu(T^{-1}(t)) \leq 1 + 3\delta, \quad \forall s, t \in V.$$

Proof. Let us fix $(V_n)_n$ be a base of X.

Case 1. $S, T \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$.

Let us define $(\Psi_\xi)_{\xi<\omega_1} \subseteq A \times X$ as follows:

$\Psi_0 = A \times X$;

for a given $\xi < \omega_1$ even, let

$\Psi_{\xi+1} = \{(x,t) : \; x \in A, \; t \in \overline{S(x) \cap \Psi_\xi(x)}\}$,

$\Psi_{\xi+2} = \{(x,t) : \; x \in A, \; t \in \overline{T(x) \cap \Psi_\xi(x)}\}$;

and for limit ordinals $\xi < \omega_1$ let

$\Psi_\xi = \bigcap_{\eta<\xi} \Psi_\eta$.

Then we have

(a) $\Psi_\xi(x)$ is closed in X, $\forall x \in A$;

(b) $\Psi_\xi \subseteq \Psi_\eta$, whenever $\eta \leq \xi < \omega_1$;

(c) $\Psi_{\xi+2}(x) \subsetneq \Psi_\xi(x)$ if $\Psi_\xi(x) \neq \emptyset$ (by our hypothesis on S and T).

Claim 1. $\Psi_\xi \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$.

Of course, for $\xi = 0$ it is clear.

Suppose the Claim holds for ξ, then

$$\begin{aligned}\Psi_{\xi+1} &= \{(x,t) : \; t \in \overline{S(x) \cap \Psi_\xi(x)}\} \\ &= \bigcap_k \{(x,t) : \text{ either } t \notin V_k \text{ or } V_k \cap S(x) \cap \Psi_\xi(x) \neq \emptyset\} \\ &= \bigcap_k \left[(A \times X \setminus V_k) \cup \pi_1(A \times V_k \cap S \cap \Psi_\xi) \times X\right].\end{aligned}$$

By the previous lemma, we have that $\Psi_{\xi+1} \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$. Similarly for $\Psi_{\xi+2}$.

Now, for all $k \in \omega$, $\xi < \omega_1$, let

$$E_{k,\xi} = \pi_1\left((A \times V_k) \cap S \cap \Psi_\xi\right)$$

Notice that, for fixed $k \in \omega$, $(E_{k,\xi})_{\xi<\omega_1}$ i a decreasing sequence in Σ.

Since μ is a probability measure,

$$\exists \eta < \omega_1 : \ \mu(E_{k,\xi}) = \mu(E_{k,\eta} \quad \forall \xi \geq \eta.$$

Set

$$A_1 = A \setminus \bigcup_{k\in\omega} (E_{k,\eta} \setminus E_{k,\eta+2}) .$$

Then $\mu(A \setminus A_1) = 0$. Let us fix $x \in A_1$, we have

$$\{k \in \omega : \ x \in E_{k,\eta+2}\} = \{k \in \omega : \ x \in E_{k,\eta}\}.$$

So

$$\Psi_{\eta+1}(x) = \{t \in X : \ \forall k \in \omega \text{ either } t \notin V_k \text{ or } x \in E_{k,\eta}\} = \Psi_{\eta+3}(x).$$

By (c) we have that $\Psi_{\eta+1}(x) = \emptyset$.

What we have is that: there is a countable ordinal $\eta_0 = \eta + 1$ and $A_1 \subseteq A$ such that

$$\mu(A \setminus A_1) = 0 \quad \text{and} \quad \Psi_{\eta_0}(x) = \emptyset, \ \forall x \in A_1.$$

Now, for all $n \in \omega$, $\xi < \omega_1$, let us define

$$\Phi_{n,\xi} = \{(x,t) \in \Psi_\xi : \ d(y, \Psi_{\xi+1}) \geq \frac{1}{2^n}\}.$$

Claim 2. $\Phi_{n,\xi} \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$.

Let $(t_n)_n$ be a countable dense subset of X. Then,

$$\Phi_{n,\xi} = \Psi_\xi \setminus \bigcup \{R(\alpha, \beta, k) : \ \alpha, \beta \in \mathbb{Q}, \ \alpha + \beta < \frac{1}{2^n}, \ k \in \omega\}$$

where

$$\begin{aligned} R(\alpha, \beta, k) &= \{(x,t) : \ d(t, t_k) \leq \beta, \ N_\alpha(t_k) \cap \Psi_{\xi+1}(x) \neq \emptyset\} \\ &= \pi_1 \left(A \times N_\alpha(t_k) \cap \Psi_{\xi+1}\right) \times N_\beta(t_k). \end{aligned}$$

and

$$N_\alpha(t_k) = \{t \in X : \ d(t, t_k) \leq \alpha\}.$$

Therefore, $\Phi_{n,\xi} \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$, for all $n \in \omega$ and $\xi < \omega_1$.

Moreover, each $\Phi_{n,\xi}$ is closed.

If $\eta < \xi$ then $d(\Phi_{n,\xi}(x), \Phi_{n,\eta}(x)) \geq \frac{1}{2^n}$ for all $n \in \omega$, $x \in A$.

Also, we have

$\Psi_{\xi+1}(x)$ is closed;

$\bigcup_{n\in\omega} \Phi_{n,\xi} = \Psi_\xi \setminus \Psi_{\xi+1}$, $\xi < \omega_1$;

$\bigcup_{n\in\omega,\eta\in\xi} \Phi_{n,\eta} = (A \times X) \setminus \Psi_\xi$, $\xi < \omega_1$.

Now, let us consider

$$\Phi_n = \bigcup_{\xi<\eta_0} \Phi_{n,\xi} \in \Sigma \overset{\wedge}{\otimes} \mathcal{B},$$

remembering that η_0 was a countable ordinal.

Let us define

$$h(x,t) = \begin{cases} 1, & \text{if } (x,t) \in \Psi_\xi \setminus \Psi_{\xi+1}, \text{ where } \xi \text{ is odd, } \xi \leq \eta_0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, h is $\Sigma \overset{\wedge}{\otimes} \mathcal{B}$ measurable. We know that, if $x \in A_1$, $t \in X$ Kipping in mind the definition of η_0, $t \notin \Psi_{\eta_0}(x)$.

Then, there exists some $\xi < \eta_0$: $t \in \Psi_\xi(x) \setminus \Psi_{\xi+1}(x)$ (by hypothesis).

If ξ is even, $h(x,t) = 0$ and $t \notin \overline{S(x) \cap \Psi_\xi(x)}$, so $(x,t) \notin S$.

If ξ is odd, then $h(x,t) = 1$ and $t \notin \overline{T(x) \cap \Psi_\xi(x)}$, so $(x,t) \notin T$.

What we have is, $\forall x \in A_1$, $t \in X$

$$\chi_S(x,t) \leq h(x,t), \qquad \chi_T(x,t) \leq 1 - h(x,t).$$

By definition of h, for any $x \in A$, $n \in \omega$, $\xi < \omega_1$, $h(x,t)$ is constant for $t \in \Phi_{n,\xi}(x)$.

Therefore, if we denote by $h_x(t) = h(x,t)$, h_x is continuous on $\Phi_n(x) = \bigcup_{\xi<\eta_0} \Phi_{n,\xi}(x)$ (because for fixed n, x, $\Phi_{n,\xi}(x)$ are isolated).

Let $B = ballC(X)$. Let us define

$$\Lambda_n = \{(x,z) \in A \times B : \ z(t) = h(x,t), \ \forall t \in \Phi_n(x)\}.$$

By Tietze's theorem (i.e., every continuous function on a closed subset of a normed space can be extendible over the whole space), $\Lambda_n(x)$ is never empty and clearly it is closed (here, for once, we are giving on B the uniform norm topology, so B is a Polish space).

Claim 3.

$$\Lambda_n : A \longrightarrow \mathcal{F}(B)$$

is a multifunction measurable; i.e., for every open V subset of B $\{x \in A : \Lambda_n(x) \cap V \neq \emptyset\} \in \Sigma$.

To show that, it is enough that $\{x \in A: \ \rho(z, \Lambda_n(x)) \leq \varepsilon\}$ is measurable, for all $z \in B, \varepsilon > 0$ (where ρ is a metric on B).

But,

$$\begin{aligned}\{x: \ \rho(z, \Lambda_n(x)) \leq \varepsilon\} &= \{x \in A: \ |z(t) - h(x,t)| \leq \varepsilon, \ \forall t \in \Phi_n(x)\} \\ &= A \setminus \pi_1 \left(\Phi_n \cap \{(x,t): \ |z(t) - h(x,t)| > \varepsilon\}\right) \in \Sigma,\end{aligned}$$

because h is $\Sigma \overset{\wedge}{\otimes} \mathcal{B}$ measurable and $\Phi_n \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$.

By the Kuratowski-Ryll-Nardzewski's theorem (see [5]),

$$\exists \lambda : A \longrightarrow B \text{ measurable function such that}$$

$$\lambda_n(x) \in \Lambda_n(x), \quad \forall x \in A.$$

Set $f_n(x,t) = \lambda_n(x)(t) : A \times X \longrightarrow \mathbb{R}$.

Then f_n is measurable in the first variable and continuous in the second one; also, $|f_n(x,t)| \leq 1$, $\forall x \in A, t \in X$.

By construction, $f_n = h$ on Φ_n. Since $X = \bigcup_{n \in \omega} \Phi_n(x)$, for $x \in A_1$, we have

$$h(x,t) = \lim_n f_n(x,t), \quad \forall x \in A_1, \ t \in X.$$

Set

$$z_n(t) = \int f_n(x,t) \ d\mu(x).$$

Then $z_n \in C_p(X)$ and

$$\lim_n z_n(t) = \int h(x,t) \ d\mu(x), \quad \forall t \in X;$$

that is because $\forall t \in X$, $\lim_n f_n(x,t) = h(x,t)$ for almost $x \in A$.

Let us consider U the open of the enunciate of the Lemma. Therefore,

$$U = \bigcup_{n \in \omega} \{t \in U : \ |z_m(t) - z_n(t)| \leq \delta, \ \forall m \geq n\}.$$

By Baire's theorem (see Theorem 1.4), there is $n_0 \in \omega$ such that

$$G = int\{t \in U : \ |z_m(t) - z_n(t)| \leq \delta, \ \forall m \geq n\} \text{ is not empty.}$$

Let $V \subseteq G$ be an open set such that $|z_n(s) - z_n(t)| \leq \delta \ \forall s, t \in V$.

Therefore,

$$|z_m(s) - z_n(t)| \leq \delta, \ \forall s, t \in V, \ m \geq n.$$

Since, for $x \in A_1$, $s, t \in V$

$$\chi_S(x,s) + \chi_T(x,t) \leq h(x,s) + 1 - h(x,t),$$

we have

$$\begin{aligned}\mu(S^{-1}(s)) + \mu(T^{-1}(t)) &\leq 1 + \int h(x,s)\ d\mu(x) - \int h(x,t)\ d\mu(x) \\ &= 1 + \lim_m [z_m(s) - z_m(t)] \leq 1 + 3\delta.\end{aligned}$$

Case 2. $S, T \subseteq A \times X$ general sets.

Suppose no such V can be found. Let $I = \{k \in \omega :\ V_k \cap U \neq \emptyset\}$.

Then we can consider, for each $k \in I$, points $s_k, t_k \in V_k \cap U$ such that

$$\mu(S^{-1}(s_k)) + \mu(T^{-1}(t_k)) > 1 + 3\delta.$$

Let

$$S_0 = \bigcup_{k \in I} S^{-1}(s_k) \times \{s_k\}, \quad T_0 = \bigcup_{k \in I} T^{-1}(t_k) \times \{t_k\}.$$

Then $T_0, S_0 \in \Sigma \overset{\wedge}{\otimes} \mathcal{B}$, $S_0 \subseteq S$ and $T_0 \subseteq T$.

By hypothesis and Case 1., $\exists F \subseteq X$, $x \in A$ such that

$$F = \overline{F \cap S_0(x)} = \overline{F \cap T_0(x)}.$$

If $V \subseteq U$ is open, then

$$\sup_{s \in V} \mu(S_0^{-1}(s)) + \sup_{t \in V} \mu(T_0^{-1}(t)) > 1 + 3\delta,$$

which clearly contradicts Case 1. □

Proposition 1.37. *Let (A, Σ, μ) be a complete probability space and X a complete metric space. Let*

$$f : A \times X \longrightarrow \mathbb{R}$$

be a bounded function, maesurable in the first variable and of 1th Baire class in the second one.

Then,

$$z(t) = \int f(x,t)\ d\mu(x) \in \mathcal{B}_1(X).$$

Proof. By Lemma 1.33, we may assume that X is a compact metric space.

By Lemma 1.34, we need to show that: $\forall\varepsilon > 0$ and non empty open $U \subseteq X$ there is a non empty open $V \subseteq U$:

$$diam(z(V)) \leq \varepsilon.$$

Since f is bounded, we can assume $0 \leq f(x,t) \leq 1$, $\forall x \in A, t \in X$.

Let $n \in \omega$ be such that $3n + 1 \leq \varepsilon n^2$. Let us set

$$S_r = \{(x,t) :\ f(x,t) \leq \frac{r}{n}\}, \quad T_r = \{(x,t) :\ f(x,t) \geq \frac{r}{n}\}.$$

For each $r \in \omega$, S_r, T_{r+1} satisfy the hypothesis of Lemma 1.36; indeed, $\forall x \in A$, the map $t \longmapsto f(x,t) \in \mathcal{B}_1(X)$ (see the proof of Lemma 1.33).

So, by induction, we can choose non empty open sets $(V_r)_r$ such that

(i) $V_0 = U$;

(ii) $V_{r+1} \subseteq V_r$;

(iii) $\mu(S_r^{-1}(s)) + \mu(T_{r+1}^{-1}(t)) \leq 1 + \frac{1}{n}$ for all $s,t \in V_{r+1}$, $0 \leq r \leq n$.

Now, $s,t \in V_{n+1}$ then

$$(1) \quad \sum_{r\leq n} \frac{1}{n}\mu(T_{r+1}^{-1}(t)) \geq \int f(x,t) - \frac{1}{n}\ d\mu(x) = z(t) - \frac{1}{n},$$

and

$$(2) \quad \sum_{r\leq n} \frac{1}{n}[1 - \mu(S_r^{-1}(s))] \leq z(s) + \frac{1}{n}.$$

To see (1), note that, since $T_{r-1}^{-1}(t) \subseteq T_r^{-1}(t)$ and $A = T_0^{-1}(t)$, we have

$$f(x,t) - \frac{1}{n} \leq 0, \quad \forall x \in T_0^{-1}(t) \setminus T_1^{-1}(t),$$

$$f(x,t) - \frac{1}{n} \leq \frac{1}{n}, \quad \forall x \in T_1^{-1}(t) \setminus T_2^{-1}(t),$$

and so on, and since

$$T_r^{-1}(t) = (T_r^{-1}(t) \setminus T_{r+1}^{-1}(t)) \cup T_{r+1}^{-1}(t),$$

we get

$$\int f(x,t) - \frac{1}{n}\ d\mu(x) = \int_{T_0^{-1}(t)\setminus T_1^{-1}(t)} f(x,t) - \frac{1}{n}\ d\mu(x) + \dots \int_{T_n^{-1}(t)\setminus T_{n+1}^{-1}(t)} f(x,t) - \frac{1}{n}\ d$$

$$\leq 0 + \frac{1}{n}\mu(T_1^{-1}(t) \setminus T_2^{-1}(t)) + \frac{2}{n}\mu(T_2^{-1}(t) \setminus T_3^{-1}(t)) + \ldots + \mu$$

$$= \frac{1}{n}\mu(T_1^{-1}(t)) + \ldots + \frac{1}{n}\mu(T_{n+1}^{-1}(t))$$

$$= \sum_{r \leq n} \mu(T_{r+1}^{-1}(t)).$$

The reader can figure out (2) similarly.

Therefore,

$$\begin{aligned} z(t) - z(s) &\leq \sum_{r\leq n} \frac{1}{n}\mu(T_{r+1}^{-1}(t)) + \frac{1}{n} - \sum_{r \leq n} \frac{1}{n}[1 - \mu(S_r^{-1}(s))] + \frac{1}{n} \\ &= \frac{2}{n} + \frac{1}{n}\sum_{r\leq n}[\mu(T_{r+1}^{-1}(t)) + \mu(S_r^{-1}(s)) - 1] \\ \text{(by } (iii)) \quad &\leq \frac{2}{n} + \frac{1}{n}(n+1)\frac{1}{n} \leq \varepsilon. \end{aligned}$$

□

Theorem 1.38. *(Talagrand)*
Let X be a complete metric space, $A \subseteq \mathcal{B}_1(X)$ a compact uniformly bounded set.

Then, $co(A)$ is relatively compact in $\mathcal{B}_1(X)$.

Proof. As in the Rosenthal's theorem 1.22, we have that $\overline{co}(A)$ is compact in $\mathbb{R}^X$.

Then, it is enough to show that $\overline{co}(A) \subseteq \mathcal{B}_1(X)$.

Let $z \in \overline{co}(A)$. As A is compact in the locally convex Hausdorff space $\mathbb{R}^X$, there is a Radon measure μ on A such that

$$f(z) = \int_A f(x)\ d\mu(x), \quad \forall f \in (\mathbb{R}^X)^*.$$

In particular

$$z(t) = \int_A x(t)\ d\mu(x), \quad \forall t \in X.$$

But, the function $h : A \times X \longrightarrow \mathbb{R}$ defined by

$$h(x,t) = x(t), \quad \forall x \in A, t \in X$$

satisfies the condition of Proposition 1.37. Hence,

$$z(t) = \int h(x,t)\ d\mu(x) \in \mathcal{B}_1(X).$$

□

1.0.2 Summertime

In this section, we are going to show how the space $C_p(X)$ and $\mathcal{B}_1(X)$ play a central rule in the Banach space theory.

Let us start with a classical result due to H. P. Rosenthal, A. Pelczynski and R. Haydon. Originally, the following result was proved using combinatorial tools. The following proof give a more topological character.

Theorem 1.39. *Let B be a separable Banach space. Then the following are equivalent*

0. *B contains a copy of ℓ_1 (i.e., ℓ_1 embeds in B);*
1. *There is a bounded sequence in B with no weak-Cauchy subsequence;*
2. *There is a bounded sequence in B^{**} with no weak*-convergent subsequence;*
3. *there is an element of B^{**} which is not 1th Baire class function on $(B_{X^*}, weak^*)$;*
4. *There is an element of B^{**} which is not weak*-limit of a sequence of B;*
5. *The cardinality of B^{**} is greater than the cardinality of B;*
6. *There is a bounded weak* strongly countably compact of B^{**} which is not weak* compact (strongly countably compact means that every separable subset has compact closure);*
7. *there is a bounded weak* closed convex subset of B^* which is not the norm closure convex hull of the set of its extreme points;*
8. *$L_1[0,1]$ embeds in B^*;*
9. *$\ell_1(\Gamma)$ embeds in B^* for some uncountable set Γ;*
10. *$C([0,1])$ is a continuous linear image of B.*

Proof. Since B is a separable Banach space, we have that $X = (B_{B^*}, weak^*)$ is a Polish space. Let us consider

$$F = \{f|_X \;\; f \in B^{**}, \;\; \|f\| \leq 1\}.$$

Therefore, F is a pointwise compact family of real-valued function on X.

(2) $\Rightarrow$ (0) Let us suppose that B^{**} have a bounded sequence with no weak* convergent subsequence. Then F fails (3) of Theorem 1.22. In particular, that implies $F \nsubseteq \mathcal{B}_1(X)$. Let $(g_n)_n \subseteq B^{**}$, $\|g_n\| \leq 1$ be such that: $(g_n)_n$ has no weak* convergent subsequence. Letting $f_n = g_n|_X$, $n \in \omega$. Then $(f_n)_n \subseteq F$ with no pointwise convergent subsequence. By Theorem 1.21, there exists $(f_{n_k})_k$ subsequence of $(f_n)_n$, $L \subseteq X$ and $f : X \longrightarrow \mathbb{R}$ such that

$$f_{n_k} \longrightarrow f \text{ pointwise}$$

$$f \text{ satisfies the Discontinuity Criterion.}$$

By the classical Goldstine 's theorem

$$(\Delta) \quad f \text{ is in the pointwise closure of } \{g|_L \ \ g \in ball(B)\}.$$

Since the elements of $ball(B)$ are continuous on L, by Proposition 1.19,

$$\ell_1 \hookrightarrow B.$$

(1) $\Rightarrow$ (0) If $g_n)_n \subseteq B$ has no weak Cauchy subsequence, then $(g_n)_n$ satisfies (Δ) above. Therefore $(g_n)_n$ has a subsequence equivalent to the usual ℓ_1-basis.

Therefore $(0) - (1) - (2)$ are equivalents.

(6) $\Rightarrow$ (0) Let us suppose (6) holds. Let F defined as above. Then F contains a strongly countable compact which is non compact Y. So Y fails the condition (a) of Theorem 1.22

$$\Rightarrow F \nsubseteq \mathcal{B}_1(X) \ \Rightarrow \ell_1 \hookrightarrow B.$$

(0) $\Rightarrow$ (6) If ℓ_1 embeds in B, then ℓ_1^{**} is weak* isomorphic to a subspace of B^{**}, and $\beta\mathbb{N}$ (the Cech-Stone compactification of $\mathbb{N}$) is homeomorphic to a weak* compact of ℓ_1^{**}.

Let us consider a family $(M_\alpha)_{\alpha<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that

$M_\alpha \cap (\mathbb{N} \setminus M_\beta)$ is infinite (for $\alpha < \beta < \omega_1$)

$M\beta \subseteq_a M_\alpha$

For any $\alpha < \omega_1$, let

$$K_\alpha = \overline{M_\alpha}^{\beta\mathbb{N}} \cap (\mathbb{N} \setminus M_\alpha).$$

Then $(K_\alpha)_{\alpha<\omega_1}$ is a family of clopen in $\beta\mathbb{N} \setminus \mathbb{N}$ with

$$K_\beta \subseteq K_\alpha, \quad \alpha < \beta < \omega_1.$$

Therefore,

$$\bigcup_{\alpha<\omega_1} (\beta\mathbb{N} \setminus K_\alpha) \cap (\beta\mathbb{N} \setminus \mathbb{N})$$

is a strongly countably compact which is non compact of $\beta\mathbb{N}$. $\square$

Before continuing to prove the all equivalences above, we need to recall the following

Definition 1.40. Let C be a convex subset of a topological vector space. A point $x_0 \in C$ is said to be an *extreme point* if $x_0 = \lambda x + (1-\lambda)y$, for some $x, y \in C$ and $\lambda \in]0,1[$, then necessarily $x_0 = x = y$. In the sequel, we shall denote by $extC$ the set of all extreme points of C.

Proposition 1.41. *If X is a metrizable compact convex subset of a topological vector space, then the extreme points of X form a G_δ set*

Proof. Suppose that the topology of X is given by the metric d. For each $n \in \omega$, let us define

$$F_n = \{x \in X : x = \frac{1}{2}y + \frac{1}{2}z,\ y, z \in X,\ d(y,z) \geq \frac{1}{n}\}.$$

It is clear that

F_n is closed, $n \in \omega$;

$x \in X$ is not an extreme point if and only if $\exists n_0 \in \omega : x \in F_{n_0}$.

Then

$$X \setminus extX = \bigcup_{n\in\omega} F_n,$$

which, of course, implies that $extX$ is a G_δ in X. □

Corollary 1.42. *If X is a complete metric space, $C \subseteq X$ is a compact convex set, then*

$extC$ is a Baire space.

Let us recall from Proposition 1.19 that:

If $(x_n)_n$ is a uniformly bounded sequence of real valued functions on a set S, $\delta, r \in \mathbb{R}$, with $\delta > 0$, and

$$A_n = \{\xi \in S :\ x_n(\xi) > \delta + r\}$$

$$B_n = \{\xi \in S :\ x_n(\xi) < r\}.$$

Assuming that $\forall F_1, F_2 \subseteq \omega$ finite and disjoints, we have

$$V(F_1,F_2) = \bigcap_{n\in F_1} A_n \cap \bigcap_{n\in F_2} B_n \neq \emptyset.$$

Then $(x_n)_n$ is equivalent (in the sup-norm) to the usual ℓ_1-basis.

Lemma 1.43. *Let B be a Banach space, S be a non empty bounded subset of B^*, $\varphi \in B^{**}$, $r, \delta \in \mathbb{R}$ with $\delta > 0$. Assume that for each weak*-open $U \subseteq B^*$ with $S \cap U \neq \emptyset$,*

$$\begin{cases} \exists \xi, \eta \in \overline{co}^{weak^*}(S \cap U) : \\ \varphi(\xi) > \delta + r, \\ \varphi(\eta) < r. \end{cases} \tag{1.1}$$

Then B contains a sequence equivalent to the usual ℓ_1-basis.

Proof. By assumption (1.1) and Golstine's theorem, $\exists x_1 \in B$ with $\|x_1\| = \|\varphi\|$ such that

$$\xi(x_1) > \delta + r, \qquad \eta(x_1) < r.$$

Since $\xi, \eta \in \overline{co}^{weak^*}(S)$, we have

$$A_1 = \{s \in S : \ s(x_1) > \delta + r\} \neq \emptyset$$

$$B_1 = \{s \in S : \ s(x_1) < r\} \neq \emptyset.$$

Suppose, by induction, $\exists x_1, \ldots, x_n \in B$ has been defined such that

$$V(F_1, F_2) \neq \emptyset, \text{ for every pair of disjoint sets } F_1, F_2 \subseteq \omega.$$

Since $V(F_1, F_2)$ is a weak* open which intersects S, by assumption, there must exist $\xi(F_1, F_2), \eta(F_1, F_2) \in \overline{co}^{weak^*}(V(F_1, F_2))$:

$$\varphi(\xi(F_1, F_2)) > \delta + r$$

$$\varphi(\eta\xi(F_1, F_2)) < r.$$

By Goldstine's theorem $\exists x_{n+1} \in B$, $\|x_{n+1}\| = \|\varphi\|$:

$$\xi(F_1, F_2)(x_{n+1}) > \delta + r$$

$$\eta(F_1, F_2)(x_{n+1}) < r$$

for every $F_1, F_2 \in \mathcal{F}_D(\omega)$.

Therefore, we have

$$A_{n+1} \cap V(F_1, F_2) \neq \emptyset, \quad B_{n+1} \cap V(F_1, F_2) \neq \emptyset, \ \forall F_1, F_2 \in \mathcal{F}_D(\omega).$$

Therefore, the lemma follows by Proposition 1.19. □

Proposition 1.44. *Let B be a Banach space such that $\ell_1 \not\hookrightarrow B$.*

Then, every weak compact convex subset of B^* is the norm closure convex hull of its extreme points.*

Proof. Let C be a weak* compact convex subset of B^* and suppose that

$$C \neq \text{ norm closure convex hull of } extC = \overline{co}^{\|\cdot\|}(extC).$$

By Hahn-Banach's theorem, there exists $\varphi \in B^{**}$ such that

$$1 = \inf\{\varphi(\xi) : \ \xi \in C\} > \sup\{\varphi(\xi) : \ \xi \in extC\}.$$

By Bishop-Phelps's theorem, we can, as well, assume that

$$F = \{\xi \in C : \ \varphi(\xi) = 1\} \neq \emptyset.$$

So, F is a norm closed face of C; let $K = \overline{F}^{weak^*}$ and $E = extK$.

Notice that $F \cap E = \emptyset$.

Indeed, if $\xi \in F \cap E$, then $\xi \in extC$ and so $\varphi(\xi) < 1$. But, E is a Baire space (see [1] or Appendix 3), then there must exists $n_0 \in \omega$ such that

$$E_{n_0} = E \cap \overline{co}^{weak^*}\{\xi \in E : \ \varphi(\xi) < 1 - \frac{1}{n_0}\}$$

contains a non empty weak* open S of E.

We claim that the lemma holds for S, $r = 1 - \frac{1}{n_0}$, $\delta = \frac{1}{2n_0}$.

Let V is a weak* open such that $V \cap S \neq \emptyset$. Since $V \cap S$ is a weak* open of E, then $\exists x \in B, \alpha \in \mathbb{R}$ such that if $W = \{\xi \in B^* : \ \xi(x) > \alpha\}$ then

$$\emptyset \neq W \cap E \subseteq V \cap S.$$

Keeping in mind that $K = \overline{F}^{weak^*}$, there must exists $\xi_0 \in W \cap F$.

If $\xi_0 \in \overline{co}^{weak^*}(W \cap E)$, we put $\xi = \xi_0$. Otherwise, there are

$$\xi_1 \in \overline{co}^{weak^*}(W \cap E), \ \xi_2 \in \overline{co}^{weak^*}(E \setminus W)$$

so that

$$\xi_0 = \lambda\xi_1 + (1-\lambda)\xi_2, \ \lambda \in [0,1].$$

Now, $\xi_2(x) \leq \alpha$, while $\xi_0(x) > \alpha$. Therefore $\lambda > 0$. Since F is a face, $\xi_1 \in F$. Then,

$$\varphi(\xi_1) = 1, \quad \xi_1 \in \overline{co}^{weak^*}(W \cap E).$$

On the other hand, $\{\eta \in S : \ \varphi(\eta) < 1 - \frac{1}{n_0}\}$ is weak* dense in S, so $V \cap S$ contains some η such that

$$\varphi(\eta) < 1 - \frac{1}{n_0}.$$

□

Proposition 1.45. *Let B be a Banach space containing a subspace isomorphic to ℓ_1. Then there is a weak* compact subset T of B^* such that*

$$\overline{co}^{weak^*}T \neq \overline{co}^{\|\cdot\|}T.$$

Proof. Let

$$j : \ell_1 \hookrightarrow B$$

be a linear homeomorphism embedding and

$$u : \ell_1 \longrightarrow C([0,1])$$

be a quotient map.

Denote by $\delta(t)$ $(t \in [0,1])$ the Dirac measure (or point mass measure). Then

$$\overline{co}^{weak^*}\{\delta(t) : \ t \in [0,1]\}$$

consists of all probability measures in $M[0,1]$, while

$$\overline{co}^{\|\cdot\|}\{\delta(t) : \ t \in [0,1]\}$$

consists just of all atomic probability measures.

Let us consider

$$S = u^*(\{\delta(t) : \ t \in [0,1]\}) \subseteq \ell_\infty.$$

Then S is weak* compact convex so that

$$\overline{co}^{weak^*}S \neq \overline{co}^{\|\cdot\|}S.$$

Finally, let $T \subseteq B^*$ be a weak* compact such that

$$j^*(T) = S.$$

Then

$$j^*(\overline{co}^{weak^*}T) = \overline{co}^{weak^*}S$$

and

$$j^*(\overline{co}^{\|\cdot\|}T) = \overline{co}^{\|\cdot\|}S.$$

That implies

$$\overline{co}^{weak^*}T \neq \overline{co}^{\|\cdot\|}T.$$

□

Note that if Γ is a uncountable abstract set, $c_0(\Gamma)$ contains no copy of ℓ_1, but it is not weak* sequentially dense in $\ell_\infty(\Gamma)$. Therefore, Theorem 1.39 above it is not true for a non separable case.

Definition 1.46. Let K be a compact Hausdorff space. A function $\varphi : K \longrightarrow \mathbb{R}$ is said to be *universally measurable* if φ is μ- measurable for every regular Borel measure μ on K. By Lusin's theorem, that means there exists, for each measure μ and $\varepsilon > 0$, a compact $L \subseteq K$ such that

$$|\mu|(K \setminus L) < \varepsilon, \quad \varphi|_L \text{ is continuous.}$$

Definition 1.47. If K is a compact convex space, $\varphi : K \longrightarrow \mathbb{R}$ satisfies the *barycentric calculus* if φ is universally measurable and

$$\int_K \varphi d\mu = \varphi(r\mu)$$

for every probability measure μ on K.

$r\mu$ is called the *resultant of* μ, defined to be the unique point of K such that

$$\int_K f d\mu = f(r\mu), \quad \text{for every continuous affine function } f \text{ on } K.$$

Proposition 1.48. *Let K be a compact convex set, $\varphi : K \longrightarrow \mathbb{R}$ be a bounded affine function. TFAE*

(i) φ satisfies the barycentric calculus;

(ii) for every probability measure μ on K, every $\varepsilon > 0$ there exists a compact convex $L \subseteq K$ with

$$\mu(L) > 1 - \varepsilon \text{ and } \varphi|_L \text{ is continuous}$$

(iii) for every $r, \delta \in \mathbb{R}$, $\delta > 0$, and every probability measure μ on K there is a closed convex $L \subseteq K$ with $\mu(L) > 0$ which is contained either in

$$A = \{\xi \in K : \ \varphi(\xi) > r\}$$

or in

$$B_\delta = \{\xi \in K : \ \varphi(\xi) < r + \delta\}.$$

Proof. $(i) \Rightarrow (ii)$ Since φ is universally measurable, given μ and $\varepsilon > 0$ there exists $S \subseteq K$ compact:

$$\mu(S) > 1 - \varepsilon \text{ and } \varphi|_S \text{ is continuous.}$$

Let $L = \overline{co}^{\|\cdot\|} S$.

Let $\mathcal{P}(S)$ be the set of all probability measures in $M(S)$ equipped with the weak* topology, and let

$$r : \mathcal{P}(S) \longrightarrow L$$

be the barycentre map. r is a continuous surjection. By hypothesis

$$\varphi \circ r(\mu) = \int_S \varphi d\mu.$$

Since φ is continuous on S, $\varphi \circ r$ is continuous on $\mathcal{P}(S)$

$$\Rightarrow \varphi \text{ is continuous on } L.$$

$(ii) \Rightarrow (iii)$ Trivial.

$(iii) \Rightarrow (i)$ Let C be a convex subset of K, μ a positive measure on K. Define

$$\mu_c(C) = \sup\{\mu(L) : \ L \text{ compact convex}, \ L \subseteq C\}.$$

Such a measure is usually called *convex inner measure* of μ.

Claim: For each probability measure μ, $\delta > 0$, then

$$\mu_c(A) + \mu_c(B_\delta) \geq 1.$$

If not, we can choose increasing sequences of compact sets $L_n \subseteq A$, $M_n \subseteq B_\delta$, with:

$$\mu_c(A) = \sup_n \mu(L_n) = \mu(\bigcup_n L_n), \quad \mu_c(B_\delta) = \sup_n \mu(M_n) = \mu(\bigcup_n M_n).$$

Let us define

$$\nu = \mu|_{K \setminus \bigcup_n (L_n \cup M_n)}.$$

If ν is not zero, by hypothesis there exists L compact : $\nu(L) > 0$ and

$$\text{either } L \subseteq A \text{ or } L \subseteq B_\delta.$$

In case $L \subseteq A$, let $L'_n = co(L \cup L_n) \subseteq A$. Then, L'_n is a compact convex such that

$$\mu(L'_n) \geq \mu(L_n) + \nu(L).$$

In such case, $\mu(L'_n) > \mu(A)$ for sufficiently large $n \in \omega$. Namely, a contradiction.

Since $K \setminus A = \bigcap_{n\in\omega} B_{\frac{1}{n}}$

$$\Rightarrow \; \mu_c(A) + \mu_c(K \setminus A) = 1.$$

In particular, A is measurable and so φ is μ-measurable.

Let us denote by $A(K)$ the Banach space of all continuous affine real-valued functions on K. Let us consider the natural embedding

$$K \hookrightarrow ball A(K)^*.$$

Therefore, we can identify φ as an element of $A(K)^{**}$.

Given $\varepsilon > 0$, let us consider $N \in \omega$ such that $\|\varphi\| \leq N\varepsilon$.

For all $-N \leq n \leq N$ let

$$C_n = \{\xi \in K : \; n\varepsilon\varphi(\xi) < (n+1)\varepsilon\}.$$

If μ is a probability measure, we have already shown that

$$\sum_{n=-N}^{N} \mu_c(C_n) = 1,$$

so there are compact convex sets $L_n \subseteq C_n$ such that

$$\sum_{-N}^{N} \mu(L_n) \geq 1 - \frac{\varepsilon}{\|\varphi\|} \tag{1.2}$$

whenever $\mu(L_n) \neq 0$. Let

$$\mu_n = \frac{1}{\mu(L_n)} \cdot \mu|_{L_n} \text{ and } \xi_n = r\mu_n,$$

otherwise, if $\mu(L_n) = 0$, we choose an arbitrary $\xi_n \in L_n \subseteq C_n$.

Therefore

$$\begin{aligned}\|r\mu - \sum_{-N}^{N} \mu(L_n)\xi_n\| &= \|r\mu - \sum_{-N}^{N} \mu(L_n) r\mu_n\| \\ &= \sup_{\substack{\|f\|\leq 1 \\ f\in A(K)}} \left| f(r\mu) - \sum_{-N}^{N} \mu(L_n) f(r\mu_n) \right|\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\substack{\|f\|\le 1\\ f\in A(K)}} \left| \int_K f d\mu - \sum_{-N}^{N} \mu(L_n) \int_{L_n} f d\frac{\mu}{\mu(L_n)} \right| \\
(L_n \text{ are disjoints}) &= \sup_{\substack{\|f\|\le 1\\ f\in A(K)}} \left| \int_K f d\mu - \int_{\cup_{-N}^{N} L_n} f d\mu \right| \\
&= \sup_{\substack{\|f\|\le 1\\ f\in A(K)}} \left| \int_{K\setminus \cup_{-N}^{N} L_n} f d\mu \right| \\
&\le \mu(K \setminus \cup_{-N}^{N} L_n) \\
&= \mu(K) - \sum_{n=-N}^{N} \mu(L_n) \\
\text{by (1.2)} &= 1 - \sum_{n=-N}^{N} \mu(L_n) \\
&\le \frac{\varepsilon}{\|\varphi\|}
\end{aligned}
$$

Therefore

$$
\left| \varphi(r\mu) - \sum_{n=-N}^{N} \mu(L_n)\varphi(\xi_n) \right| \le \varepsilon. \tag{1.3}
$$

Since $\xi_n \in C_n$, we get

$$
\begin{aligned}
\left| \varphi(r\mu) - \sum_{n=-N}^{N} \mu(L_n) n\varepsilon \right| &\le \left| \varphi(r\mu) - \sum_{n=-N}^{N} \mu(L_n)\varphi(\xi_n) \right| \\
&+ \left| \sum_{n=-N}^{N} \mu(L_n)\varphi(\xi_n) - \sum_{n=-N}^{N} \mu(L_n) n\varepsilon \right| \\
&\le \varepsilon + \left| \sum_{n=-N}^{N} \mu(L_n)(\varphi(\xi_n - n\varepsilon) \right| \\
&= \varepsilon + \sum_{n=-N}^{N} \mu(L_n)(\varphi(\xi_n - n\varepsilon) \\
&\le \varepsilon + \mu(\bigcup_{n=-N}^{N} L_n) \\
&\le 2\varepsilon.
\end{aligned}
$$

On the other hand, by (1.3)

$$\left|\int_K \varphi d\mu - \sum_{n=-N}^{N} \int_{L_n} \varphi d\mu\right| \leq \varepsilon$$

$$\Rightarrow \left|\int_K \varphi d\mu - \sum_{n=-N}^{N} \mu(L_n)\cdot n\varepsilon\right| \leq 2\varepsilon.$$

Thus

$$\left|\varphi(r\mu) - \int_K \varphi d\mu\right| \leq 4\varepsilon.$$

□

Theorem 1.49. *(R. Haydon)*
Let B be a Banach space and $K = (ball B^*, weak^*)$. TFAE

(i) B contains no copy of ℓ_1;

(ii) every element of B^{**} is universally measurable as functions on K;

(iii) every element of B^{**} satisfies the barycentric calculus on K.

Proof. $(i) \Rightarrow (ii)$ Let μ be a probability measure on K, $\varphi \in B^{**}$, $r, \delta \in \mathbb{R}$ with $\delta > 0$. Let $S = supp\mu$. By Lemma 1.43 there is a weak* open V with $S \cap V \neq \emptyset$ so that

$$\text{either } \overline{co}^{weak^*} S \cap V \subseteq \{\xi \in K:\ \varphi(\xi) > r\}$$

$$\text{or } \overline{co}^{weak^*} S \cap V \subseteq \{\xi \in K:\ \varphi(\xi) < r + \delta\}.$$

We have that $\mu(S\cap V) > 0$. Thus (iii) of the previous proposition holds with $L = \overline{co}^{weak^*} S \cap V$.

$(iii) \Rightarrow (ii)$ Trivial.

$(ii) \Rightarrow (i)$ Let us suppose that B contains a copy of ℓ_1, let

$$j : \ell_1 \hookrightarrow B$$

be an embedding with $\|j\| = 1$. Let λ be the product measure on $\{-1, 1\}^\omega$ with $\lambda \in \ell_\infty$. Finally, let μ be a measure on K such that $j^*\mu = \lambda$.

Since $\ell_\infty = C(\beta\mathbb{N})$, then $\beta\mathbb{N} \hookrightarrow \ell_\infty^*$.

Choose $\chi \in \beta\mathbb{N} \setminus \mathbb{N}$ and consider $\varphi = j^{**}\chi$.

It is known that χ is not λ- measurable. Therefore, φ is not μ-measurable.

□

Chapter 2

Appendix

2.0.3 Appendix 1

Theorem 2.1. *Let (X, d) be a metric space and μ be a Borel probability measure on X. Then given a Borel set $B \subseteq X$ and $\varepsilon > 0$ there is a closed set $F \subseteq B$ and an open set $G \supseteq B$ such that*

$$\mu(G \setminus F) < \varepsilon. \tag{2.1}$$

Proof. Suppose $C \subseteq X$ is a non empty closed set. Let $f(x) = d(x, C)$. Then, f is continuous and $C = \{x \in X : \ f(x) = 0\}$. Let

$$C_n = \{x \in X : \ f(x) < \frac{1}{n}\}.$$

For each $n \in \omega$, C_n is an open set with $C_n \supseteq C$ and such that $\mu(C_n) \searrow \mu(C)$. Therefore, every closed satisfies (2.1).

Let $\mathcal{B}$ the family of Borel set which satisfy (2.1).

First notice that, if $(B_n)_n \subseteq \mathcal{B}$ then $\bigcup_n B_n \in \mathcal{B}$.

Indeed, fixing $\varepsilon > 0$ we can pick $F_n \subseteq B_n$ a closed, $G_n \supseteq B_n$ an open such that $\mu(G_n \setminus F_n) < \frac{\varepsilon}{2^{n+1}}$. Let us consider n_0 such that

$$\mu(\bigcup_n F_n \setminus \bigcup_{k=1}^{n_0} F_k) < \varepsilon.$$

Therefore we have, $\bigcup_{k=1}^{n_0} F_k$ is a closed set, $\bigcup_{n \in \omega} G_n$ is open with

$$\bigcup_{k=1}^{n_0} F_k \subseteq \bigcup_{n \in \omega} B_n \subseteq \bigcup_{n \in \omega} G_n,$$

and

$$\mu(\bigcup_{n\in\omega} G_n \setminus \bigcup_{k=1}^{n_0} F_k) < \varepsilon.$$

So, $\mathcal{B}$ contains the smallest σ-algebra generated by open sets. $\square$

Theorem 2.2. *(Ulam)*
Let X be a Polish space and μ be a Borel probability measure on X. Then given a Borel set B, $\varepsilon > 0$, there is a compact set $K \subseteq B$ such that

$$\mu(B \setminus K) < \varepsilon.$$

Proof. It is enough to show that there is a compact set K such that

$$\mu(K) > 1 - \varepsilon.$$

Since X is separable, for each $n \in \omega$ there is a family $(B_k(n))_k$ of balls of X such that

$$X = \bigcup_k B_k(n), \quad diam(B_k(n)) \leq \frac{1}{n}.$$

Without loose in generality, we can assume that the centers of $(B_k(n))_k$ coincide with those of $(B_k(m))_k$. Then

$$\mu(X \setminus \bigcup_{i=1}^{k(1)} B_i(1)) < \frac{\varepsilon}{2},$$

$$\mu(X \setminus \bigcup_{i=1}^{k(2)} B_i(2)) < \frac{\varepsilon}{2^2},$$

and so on.

Conclusion:

$$\bigcap_{n\in\omega}(B_1(n) \cup \ldots \cup B_{k(n)}(n))$$

is totally bounded, and

$$K = \overline{\bigcap_{n\in\omega}(B_1(n) \cup \ldots \cup B_{k(n)}(n))}$$

is compact. By construction

$$\mu(K) > 1 - \varepsilon.$$

$\square$

2.0.4 Appendix 2

Let E be a locally convex space and let $X \subseteq E$ be a compact convex subset.

Definition 2.3. A real valued function h defined on X is called *affine* if

$$h(\lambda x + (1-\lambda)y) = \lambda h(x) + (1-\lambda)h(y), \quad \forall x, y \in X, \ \lambda \in [0,1].$$

Remark 2.4. Notice that not all all affine function is of the form $x \longmapsto f(x) + r$, for some $f \in E^*$, $r \in \mathbb{R}$.

Indeed, consider $E = (\ell_2, weak)$ and $X == \{(x_n)_n \in E : \ |x_n| \leq \frac{1}{2^n}\}$. Define

$$f : X \longrightarrow \mathbb{R} \quad \text{by } f(x) = \sum_n x_n$$

Then, f is affine with $f(0) = 0$. But the is no point $y \in \ell_2$ such that $f(x) = \langle x, y \rangle$.

Consider $\mathcal{A}$ the uniformly closed subspace of $C(X)$ consisting of all real valued affine functions on X, and let

$$M = E^*|_X + \mathbb{R}.$$

The remark above says us that M is a proper subspace of $\mathcal{A}$

Proposition 2.5. *The subspace M is uniformly dense in the closed subspace $\mathcal{A}$ of all affine continuous functions on X.*

Proof. Suppose $g \in \mathcal{A}$, $\varepsilon > 0$. Let us consider the following subset of $E \times \mathbb{R}$

$$J_1 = \{(x, r) : \ r = g(x)\}$$

$$J_2 = \{(x, r) : \ r = g(x) + \varepsilon\}.$$

Those sets are compact, convex , non empty and disjoints.

Using Hahn-Banach separation to 0 and $J_2 - J_1$, the exists a continuous linear functional L on $E \times \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that

$$\sup L(J_1) < \lambda < \inf L(J_2).$$

Let f be the function on E defined by the equation $L(x, f(x)) = \lambda$.

It is clear that f is affine and continuous. Moreover,

$$g(x) < f(x) < g(x) + \varepsilon \quad \forall x \in X,$$

and $f \in M$. Notice that, the fact that f is affine on E, implies that $f = h + r$, $h \in E^*$, $r \in \mathbb{R}$. Therefore,

$$\widetilde{f} = f|_X = h|_X + r \;\Rightarrow\; \widetilde{f} \in M.$$

Let us say some more about f:

For each $x \in X$ there exists unique $r_x \in \mathbb{R}$ such that $L(x, r_x) = \lambda$.

Indeed, suppose there are $r_1, r_2 \in \mathbb{R}$ so that

$$L(x, r_1) = \lambda = L(x, r_2).$$

Then, $L(0, r_1 - r_2) = 0$, or $(r_1 - r_2)L(0, 1) = 0$, which implies $r_1 = r_2$.

That shows f is well defined. Moreover f is affine. Indeed,

$$L(tx + (1-t)y, tf(x) + (1-t)f(y)) = tL(x, f(x)) + (1-t)L(y, f(y)) = \lambda$$

$$\Rightarrow\; f(tx + (1-t)y) = f(x)) + (1-t)L(y, f(y).$$

Finally, let us shows that f is continuous.

If $x_n \to x$, and $|f(x_n) - f(x)| > \delta > 0$. Since

$$L(x_n, f(x_n)) = \lambda = L(x, f(x))$$

we get

$$0 = L(x_n - x, f(x_n) - f(x)) = (f(x_n) - f(x))L(\frac{x_n - x}{f(x_n) - f(x)}, 1).$$

Since $(\frac{1}{f(x_n) - f(x)})_n$ is a bounded sequence in $\mathbb{R}$, we have

$$\frac{x_n - x}{f(x_n) - f(x)} \longrightarrow 0.$$

Therefore, since L is continuous

$$L(\frac{x_n - x}{f(x_n) - f(x)}, 1) \longrightarrow 1.$$

Thus $f(x_n) \to f(x)$. Namely a contradiction. □

2.0.5 Appendix 3

Let E be a topological space and α, β two players, with β the first to move. The game is:

each player chooses a non empty set V in E lying inn the opponent's previously chosen set.

The space E is called *α-favorable* if α has a winning tactic no matter what β chooses, i.e. α can choose sets V_n such that $\bigcap_n V_n \neq \emptyset$. A mathematical definition can be

Definition 2.6. Let (E, θ) be a topological space. We say that E is *α-favorable* iff there is a map $f : \theta \longrightarrow \theta$ such that

$f(U) \subseteq U$ for all $U \in \theta$,

for any sequence $V_1, V_3, \ldots, V_{2n+1}, \ldots$ so that

$$V_1 \supseteq f(V_1) \supseteq V_3 \supseteq f(V_3) \supseteq \ldots$$

we have

$$\bigcap_{n \in \omega} V_n \neq \emptyset.$$

Example 2.7. (i) Every complete metric space is α-favorable.

Indeed, define a function f such that

$$diam f(U) \leq \frac{1}{2} \inf\{1, diam U\}$$

and

$$f(U) \subseteq \overline{f(U)} \subseteq U.$$

Then given $V_1, V_3, \ldots, V_{2n+1}, \ldots : V_1 \supseteq f(V_1) \supseteq V_3 \supseteq f(V_3) \supseteq \ldots$ consider $x_n \in V_n$, $n \in \omega$. Then $(x_n)_n$ is Cauchy and the limit

$$x = \lim_n x_n \in \bigcap_n V_n.$$

(ii) Every locally compact Hausdorff space is α-favorable.

In such case, choose $f(U)$ with $\overline{f(U)}$ compact and $\overline{f(U)} \subseteq U$. Then by Cantor's theorem, if V_n are as in the definition, we get

$$\bigcap_{n \in \omega} V_n \neq \emptyset.$$

Theorem 2.8. *Every α-favorable topological space is a Baire space.*

Proof. Suppose E is not Baire, then there are closed nowhere dense sets F_n such that

$$int(\bigcup_{n\in\omega} F_n) \supseteq V$$

for some non empty open set V.

Let $V_1 = V$ and $V_{2n+1} = V_{2n-1} \cap (E \setminus F_n)$.

Then there is no f giving a winning strategy since

$$V \cap (E \setminus \bigcup_n F_n) = \emptyset.$$

□

Lemma 2.9. *Let E be a Hausdorff TVS, $X \subseteq E$ convex and $A \subseteq X$ a convex linearly compact (i.e., any line intersecting A does so in a closed segment).*

Suppose $X \setminus A = B$ is convex. Then if $ext(A) \neq \emptyset$ we have

$$ext(A) \cap ext(X) \neq \emptyset.$$

Proof. Let $a \in ext(A)$ and suppose $ext(A) \cap ext(X) = \emptyset$. Therefore $a \notin ext(X)$. Then

$$a = \frac{1}{2}x + \frac{1}{2}y, \text{ for some } x \neq y \text{ in } X.$$

Since A, B are convex, we can suppose that $x \in A, y \in B$. Let $\ell = line\{x, y\}$. By hypothesis $\ell \cap A = [a, b]$, $b \in A$ (because $a \in ext(A)$).

Claim: $b \in ext(X)$.

Suppose not, then $b = \frac{1}{2}b_1 + \frac{1}{2}b_2$, $b_1 \neq b_2$ with $b_1 \in A$. Let $\ell' = line\{b_1, b_2\}$.

By construction, $b_1 \notin \ell$ (since $\ell \cap A = [a, b]$).

For $c_1, c_2 \in co\{b_1, b_2, y\}$ lying in separate open half space $y - b$, let

$$g(c_1, c_2) = \lambda c_1 + (1 - \lambda)c_2$$

so that $g(c_1, c_2) \in span(y, b)$.

Subclaim: We can choose $b_2 \in A$.

Suppose not, the we can find $z_n \in]b, b_2] \cap B$ such that $z_n \to b$. Then $\forall z \in [b_1, y[$

$$g(z, z_n) \to b.$$

If $z \in [b_1, y[$ then $g(z, z_n) \in B$ (since B is convex).

Then $[b_1, y[\subseteq A$. Since A is linearly compact, we get $[b_1, y[\subseteq A$. Namely a contradiction, because $y \in B$.

Then we can assume $b_2 \in A$. Let c_i be the end point of the segment $[b_i, y] \cap A$, $i = 1, 2$. Then $c_i \neq y$, $i = 1, 2$. Therefore we can choose

$$d_n^i \in [b_i, y] \cap B$$

such that

$$d_n^i \longrightarrow c_i, \; i = 1, 2.$$

Let $e_n = g(d_n^1, d_n^2) \in B$.

Then $e_n \to g(c_1, c_2) \in A$. It follows that $g(c_1, c_2) = a$. Or $a \notin ext(A)$. A contradiction. □

Theorem 2.10. *(Choquet)*
Let E be a Hausdorff LCS and $X \subseteq E$ be a convex compact subset.

Then $ext(X)$ is α-favorable. In particular, $ext(X)$ is a Baire space.

Proof. Given an open set $A \subseteq ext(X)$, and $a \in A$ we can choose a closed slide V of X such that

$$V \cap ext(X) \subseteq A.$$

Slide means a set of type: $\exists x^* \in E^*$, $V = X \cap \{x \in E : \; x^*(x) \leq r\}$ for some $r \in \mathbb{R}$.

Define

$$\varphi(A, a) = V \cap ext(X)$$

Of course, we can assume that $\varphi(A_1, a_1) \subseteq \varphi(A_2, a_2)$ whenever $A_1 \subseteq A_2$.

If $V_1, V_2, \ldots$ is a decreasing sequence of closed slides of X corresponding to $A_1, A_2, \ldots$, since X is compact we get

$$\bigcap_n V_n \neq .$$

But $\bigcap_n V_n$ is convex, closed set and $X \setminus \bigcap_n V_n$ is convex in X. Then, by the previous lemma, we have

$$\bigcap_n V_n \cap ex(X) \neq \emptyset$$

$$\Rightarrow \bigcap_n A_n \neq \emptyset.$$

□

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